Non Locally-Connected Julia Sets constructed by iterated tuning

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Notations:

Every quadratic map

$$f_c(z) \;=\; z^2 + c$$

has two fixed points, α and β, where

$$\alpha + \beta \;=\; 1\,, \qquad \alpha\beta = c\,, \qquad \Re(\alpha) \le \Re(\beta)\,.$$

The multiplier

$$\mu \;=\; f'_c(\alpha) \;=\; 2\alpha$$

will be used as an alternative parameter for the quadratic family.

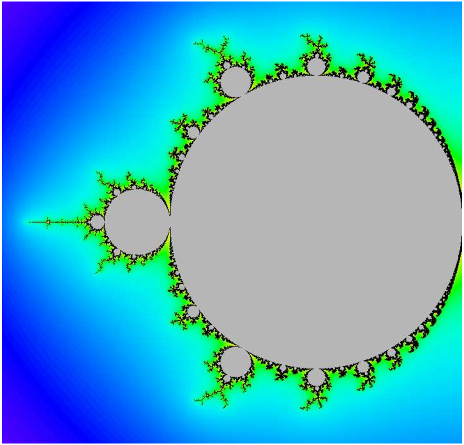
Here c and μ determine each other:

$$c \;=\; c(\mu) \;=\; \alpha(1-\alpha) \qquad \text{with} \qquad \alpha = \mu/2\,,$$

$$\mu \;=\mu(c) \;=\; 1 - \sqrt{1-4c}\,, \quad \text{with} \;\; \Re(\mu) \le 1\,.$$

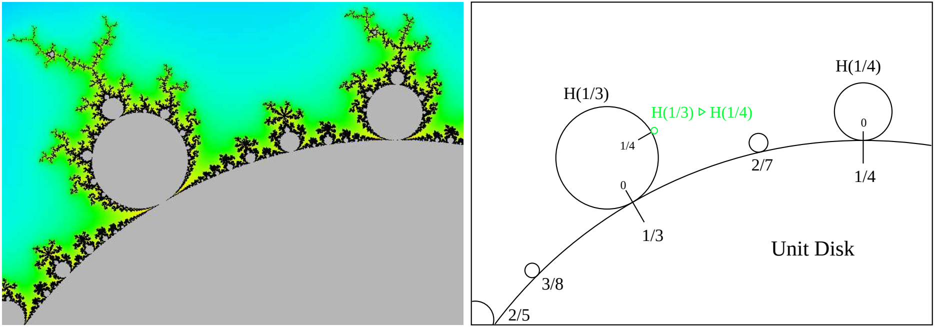
The map $f_{c(\mu)}$ corresponding to μ will be denoted by

$$\widehat{f}_\mu(z) \;=\; z^2 + c(\mu)\,.$$



The ***connectedness locus*** $\widehat{M}$, consisting of all μ in the half-plane $\Re(\mu) \le 1$ with $K(\widehat{f}_\mu)$ connected, will be called the ***rounded Mandelbrot set***.

> *Its* ***period one hyperbolic component****, the set of all $\mu \in \widehat{M}$ for which $\widehat{f}_\mu$ has an attracting fixed point, is the open unit disk $\mathbb{D}$.*



Region in $\widehat{M}$ and Disk Approximation

There is a ***satellite*** hyperbolic component $H(n/p)$ of period p attached to $\mathbb{D}$ at each p-th root of unity $e^{2\pi in/p}$.

Similarly, there are satellites $H(n/p) \triangleright H(n'/p')$ *of period* pp' *attached to* $H(n/p)$ *at corresponding boundary points; and so on.*

Empirically, each iterated satellite $H(n_1/p_1) \triangleright \cdots \triangleright H(n_k/p_k)$ can be approximated by a round disk of radius $1/(p_1 \cdots p_k)^2$.

Question: How can this be made precise?

Tuning (in parameter space).

The Douady-Hubbard ***tuning*** construction assigns to each hyperbolic component $H \subset \widehat{M}$ a homeomorphism

$$H\triangleright \; : \; \widehat{M} \xrightarrow{\cong} (H \triangleright \widehat{M}) \subset \widehat{M}$$

from $\widehat{M}$ onto a "small copy" of $\widehat{M}$.

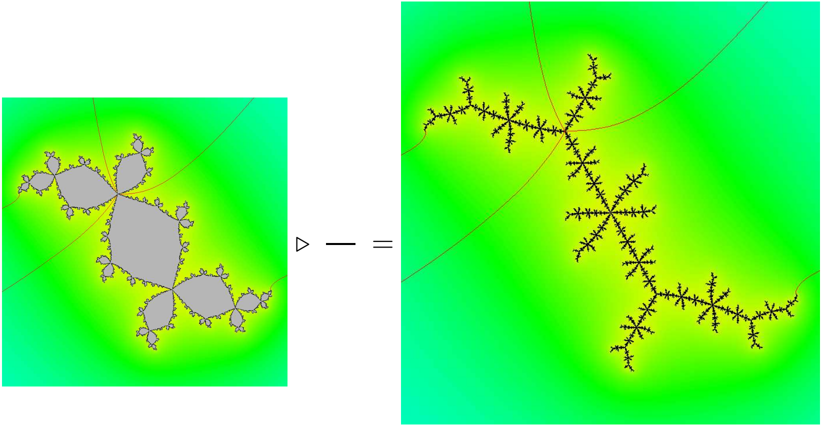
• Each $H\triangleright$ maps hyperbolic components to hyperbolic components, with $\mathrm{per}(H \triangleright H') = \mathrm{per}(H)\,\mathrm{per}(H')$.

• The set of all H forms a free non-commutative monoid, with $\mathbb{D}$ as identity element.

• Each $H\triangleright : \overline{\mathbb{D}} \to \overline{H}$ is holomorphic, and yields the canonical ***Douady-Hubbard parametrization*** of H: For each $\mu \in \mathbb{D}$, the attracting periodic orbit for the map $\widehat{f}_{H \triangleright \mu}$ has multiplier μ.

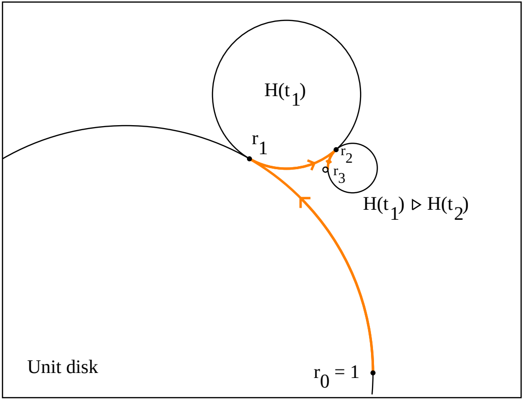
• The image $H \triangleright 1 \in \partial H$ is called the ***root point*** of H.

Tuning in the dynamic plane (intuitive picture).



To obtain the filled Julia set for $H_1 \triangleright \mu_2$, choose $\mu_1 \in H_1$, then replace every Fatou component of $K(\widehat{f}_{\mu_1})$ by a copy of $K(\widehat{f}_{\mu_2})$. (In the figure, $\widehat{f}_{\mu_2}$ is the Chebyshev map $z \mapsto z^2 - 2$, and $K(\widehat{f}_{\mu_2})$ is the line segment $[-2, 2]$.)

Constructing a non locally-connected $K(\widehat{f_\mu})$



(Small disk sizes exaggerated.)

Choose any sequence of rational angles $t_1, t_2, \ldots \neq 0$, and let r_k be the root point of

$$H(t_1) \triangleright \cdots \triangleright H(t_k).$$

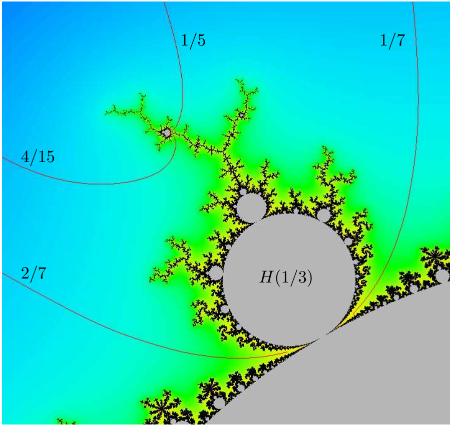
The Theorem.

Let $\omega \in \widehat{M}$ be any limit point for the sequence of root points

$$r_k \in \partial\Big(H(t_1) \triangleright \cdots \triangleright H(t_k)\Big) \qquad \text{as} \qquad k \to \infty\,.$$

Theorem (Douady, Hubbard, Sørensen).
If the sequence $\{|t_j|\}$ converges to zero sufficiently rapidly, then the filled Julia set $K(\widehat{f}_\omega)$ is not locally-connected.

The proof will be based on ***external rays*** and ***separating periodic orbits.***

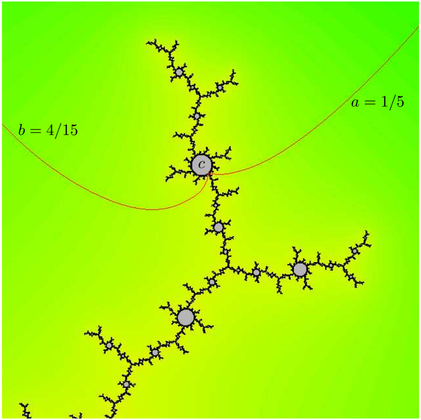


Two nested wakes $W_{[1/7,\,2/7]} \supset W_{[1/5,\,4/15]}$.

Every root point $H \triangleright 1 \in \widehat{M}$ is the landing point of exactly two external rays, with angles $0 \le a < b \le 1$. These rays cut the parameter plane into two halves.

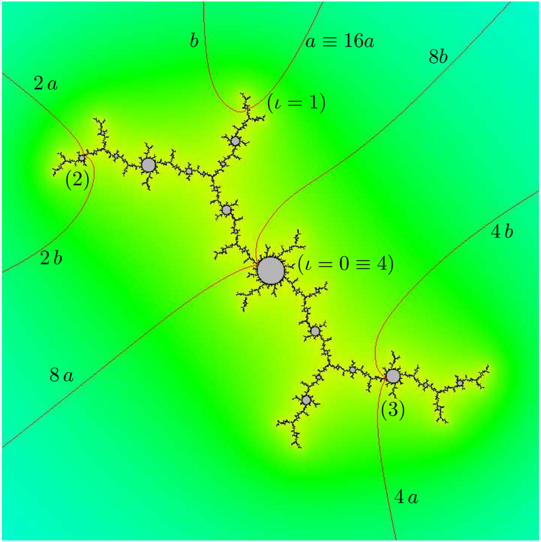
Definition. The half containing $H = H_{[a,b]}$ is called the ***wake*** $W_{[a,b]}$, and $[a, b]$ is called its ***characteristic interval***.

In the Dynamic Plane:



For every hyperbolic component $H = H_{[a,b]}$ of period $p > 1$ and every $\mu \in H$, the external rays of angle a and b for $K(\widehat{f}_\mu)$ land at a common repelling periodic point. I will write

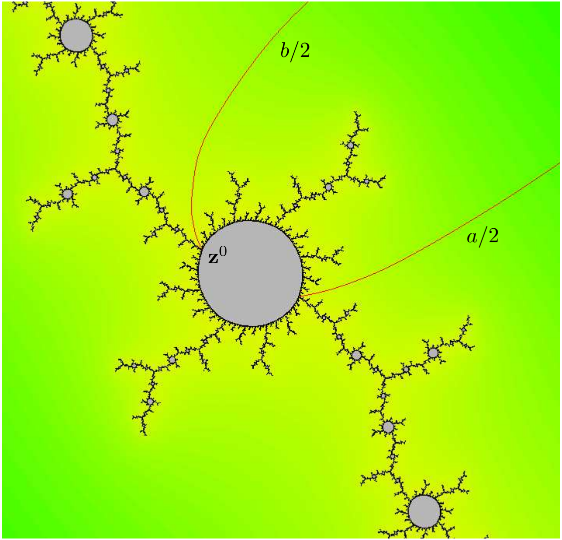
$$z^1 = z^1([a,b], \mu) = \ell_a(\mu) = \ell_b(\mu) \in \partial K(\widehat{f}_\mu).$$



Number the periodic Fatou components of $\widehat{f}_\mu$ as $0 \in U_0 \mapsto U_1 \mapsto \cdots$. Then the orbit of z^1 consists of points

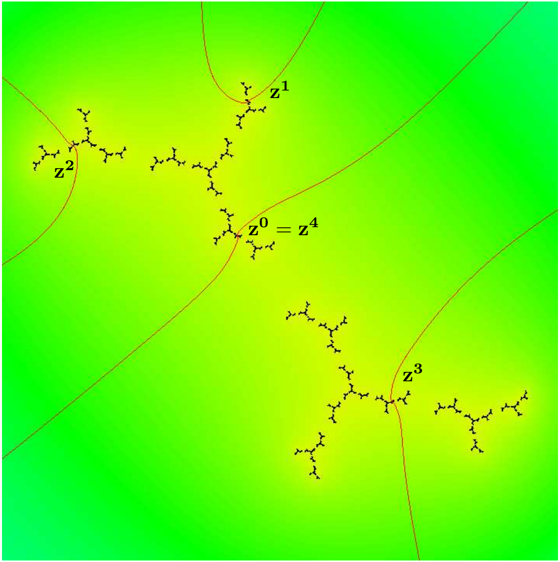
$$z^\iota = z^\iota([a,b],\,\mu) = \ell_{2^{\iota-1}a}(\mu) = \ell_{2^{\iota-1}b}(\mu) \in \partial U_\iota$$

called ***dynamic root points***, indexed by $\iota \in \mathbb{Z}/p$.



The $a/2$ and $b/2$ rays land on *opposite* sides of U_0 :

$$\ell_{a/2}(\mu) \;=\; \pm z^0([a,b],\,\mu) \quad \text{and} \quad \ell_{b/2}(\mu) \;=\; \mp z^0([a,b],\,\mu)\,.$$



More generally, these periodic points $z^\iota([a,b],\,\mu) \in \partial K(\widehat{f}_\mu)$ are defined, and vary holomorphically with μ, for all μ in the wake $W_{[a,b]}$, even when $\mu \notin \widehat{M}$.

Given any infinite sequence of hyperbolic components H_k of period $p_k > 1$, any point in the nested intersection,

$$\omega \in \bigcap_k (H_1 \triangleright \cdots \triangleright H_k \triangleright \widehat{M}) ,$$

is said to be ***infinitely renormalizable***. Let $[a_k, b_k]$ be the characteristic interval for the k-fold tuning product

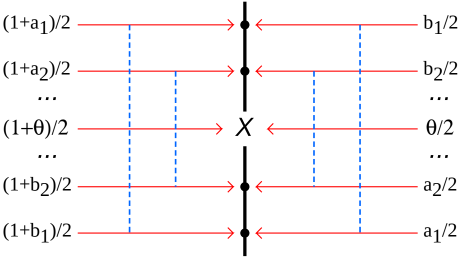
$$H_k^{\circledast} = H_1 \triangleright \cdots \triangleright H_k .$$

I will consider only the case where the nested intersection $\bigcap [a_k, b_k]$ consists of a single angle θ. (An equivalent condition would be that $H_k \cap \mathbb{R} = \emptyset$ for infinitely many k.)

Note that the points

$$\mathbf{z}_k(\omega) = z^0([a_k, b_k], \omega)$$

and their negatives cut the filled Julia set $K\left(\widehat{f}_\omega\right)$ into countably many pieces.



In this schematic diagram, externals rays are orange, equipotentials are blue, and the Julia set is black.

Let X be the connected component of zero in the set

$$K\left(\widehat{f}_\omega\right) \setminus \{\pm \mathbf{z}_k(\omega)\}\,.$$

Lemma 1. *X is compact, connected, and cellular. Every limit point of $\{\pm \mathbf{z}_k(\omega)\}$, and every limit point of the $\theta/2$ and $(1+\theta)/2$ rays, belongs to X. But every other ray is bounded away from X. If $K(\widehat{f}_\omega)$ is locally connected then $X = \{0\}$.*

Conjecture. Conversely, if $X = \{0\}$, then $K(\widehat{f}_\omega)$ is locally connected.

Choosing the Angles.

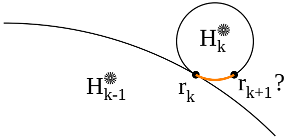
Now assume that the H_k are satellite hyperbolic components $H(n_k/p_k)$. Again let $H_k^{\circledast} = H_1 \triangleright H_2 \triangleright \cdots \triangleright H_k$, with characteristic interval $[a_k, b_k]$, and let r_k be the root point of $H_k^{\circledast}$. Then the periodic point

$$\mathbf{z}_j(r_k) = z^0\big([a_j, b_j], r_k\big) \in \partial K(\widehat{f}_{r_k}) .$$

is defined for all $j \le k$. (This point has period $p_1 \cdots p_{j-1}$. It is parabolic for $j = k$, and repelling for $j < k$.)

> **Lemma 2.** *We can choose the angles* $t_j = n_j/p_j \neq 0$ *inductively so that these points* $\mathbf{z}_j(r_k)$ *are within some specified neighborhood of* $\mathbf{z}_1(r_1)$ *for all* $j \le k$.

Start Proof. Suppose $H_1, \ldots, H_k$ have already been chosen. We must show that each $\mathbf{z}_j(r_{k+1})$ with $j \le k+1$ depends continuously on the choice of r_{k+1}, and hence can be placed arbitrarily close to $\mathbf{z}_j(r_k)$ by choosing r_{k+1} close to r_k.



Since $r_1, \ldots, r_k$ have been chosen, $\mathbf{z}_k(r_k)$ is a well defined parabolic point of period $p_1 \cdots p_{k-1}$ and multiplier $e^{2\pi i n_k/p_k}$.

For μ in a small neighborhood of r_k, the orbit of $\mathbf{z}_k(r_k)$ splits into an orbit of the same period $p_1 \cdots p_{k-1}$ with multiplier $\approx e^{2\pi i n_k/p_k}$, and a nearby orbit of period $p_1 \cdots p_k$ with multiplier $\approx +1$.

Take $\mu = r_{k+1}$ to be a point at rational angle along the boundary of H_k^*. Then this new orbit will again be parabolic, and the orbit point $\mathbf{z}_{k+1}(r_{k+1})$ will converge to $\mathbf{z}_k(r_k)$ as r_{k+1} converges to r_k.

Since the points $\mathbf{z}_j(r_{k+1})$ with $j \leq k$ clearly vary continuously with r_{k+1}, this proves Lemma 2. $\square$

Proof of the Theorem.

Recall that:

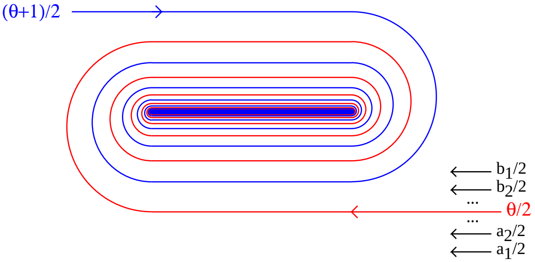
- The sequence $\{r_k\}$ of root points has ω as limit point.
- The function $\mu \mapsto \mathbf{z}_j(\mu)$ is continuous for $\mu \in W_{[a_j, b_j]}$.

Therefore, for each fixed j, the sequence of points $\mathbf{z}_j(r_k)$ has $\mathbf{z}_j(\omega)$ as a limit point.

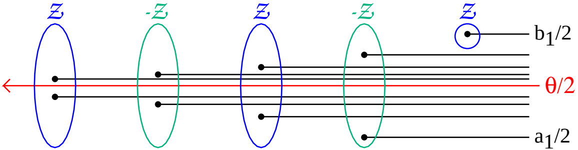
By Lemma 2, we can choose the H_k so that the $\mathbf{z}_j(r_k)$ are uniformly bounded away from 0.

Hence the points $\mathbf{z}_j(\omega)$ are also bounded away from 0.

Therefore, by Lemma 1, $K(\widehat{f_\omega})$ cannot be locally connected. □

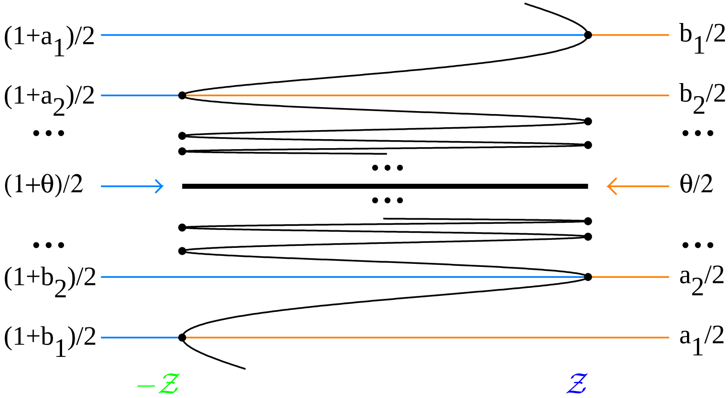


If $t_k > 0$ for all angles t_k, then the $\theta/2$ and $(\theta+1)/2$ rays spiral around each other without landing, in a “paper clip” pattern as sketched above, with the Julia set spiraling between them.



Here is a schematic picture close to the $(\theta/2)$-ray, which has been straightened out. All the points $\mathbf{z}_j(\omega)$ are assumed to lie in the region $\mathcal{Z}$, while their negatives lie in $-\mathcal{Z}$.

The $sin(1/x)$ model.



On the hand, if the signs of the t_k alternate, then the Julia set (indicated here in black) contains a $\sin(1/x)$-like curve. Compare Sørensen.

How rapidly must $t_k \to 0$?

What rate of convergence is needed to guarantee that the $\mathbf{z}_k(\omega)$ do **not** converge to zero? Here is a wild guess.
Perhaps there are order of magnitude estimates something like the following

$$\log \frac{\mathbf{z}_k(\omega)}{\mathbf{z}_{k+1}(\omega)} \approx \log \frac{\mathbf{z}_k(r_k)}{\mathbf{z}_{k+1}(r_{k+1})} \approx t_{k+1}^{1/p_k}$$

so that $\{\mathbf{z}_k(\omega)\}$ converges to zero if and only if

$$\sum_k t_{k+1}^{1/p_k} = \infty . \qquad (??)$$

For example, if $t_k = 1/p_k$ with $p_{k+1} = (k+1)^{p_k}$, then

$p_1 = 1, \; p_2 = 2, \; p_3 = 9, \; p_4 = 262144, \; p_5 \approx 1.2 \times 10^{27}, \; \ldots$

tending rapidly to infinity. Yet $t_{k+1}^{1/p_k} = 1/(k+1)$ with sum $+\infty$.

Conjecturally, this $\{p_k\}$ does not increase fast enough!

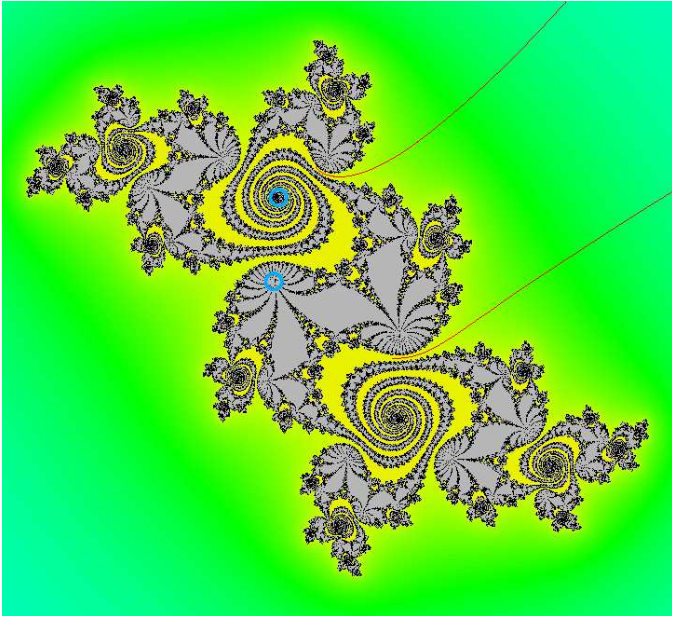
To conclude: Four Pictures

It is probably impossible to make any real picture of one of these non locally-connected Julia sets. However, we may get some intuitive idea by looking at relatively modest iterated satellite tunings.

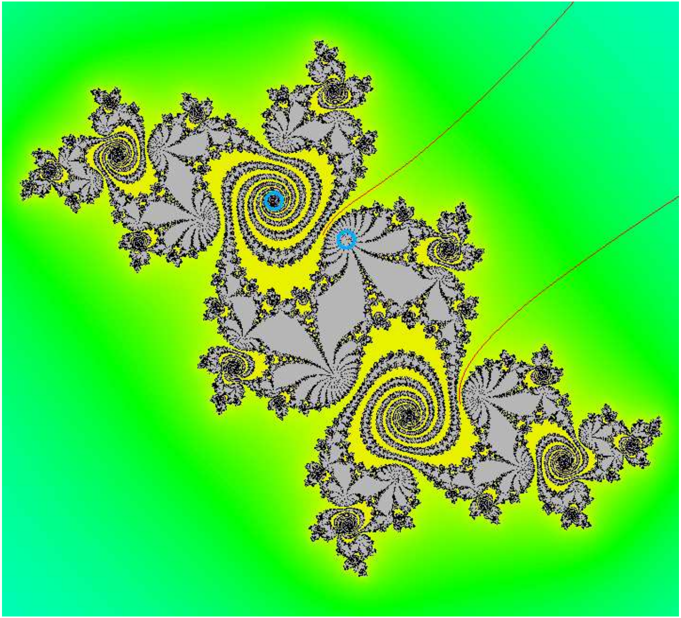
In the first two pictures, the separating periodic points $\mathbf{z}_1$ and $\mathbf{z}_2$ are circled. The rays of angle $a_1/2 = 1/14$ and $b_1/2 = 1/7$ are shown, but those of angle $a_2/2 \approx b_2/2$ are too close to distinguish from $1/14$ respectively $1/7$.

In the last two pictures, $\mathbf{z}_1, \mathbf{z}_2$ and $\mathbf{z}_3$ are defined and circled. (As the angles t_2, t_3 tend to zero, these circled points would converge towards each other.) In these cases, the rays of angle $a_1/2 = 1/14 < a_2/2 < b_2/2 < b_1/2 = 1/7$ can be distinguished.

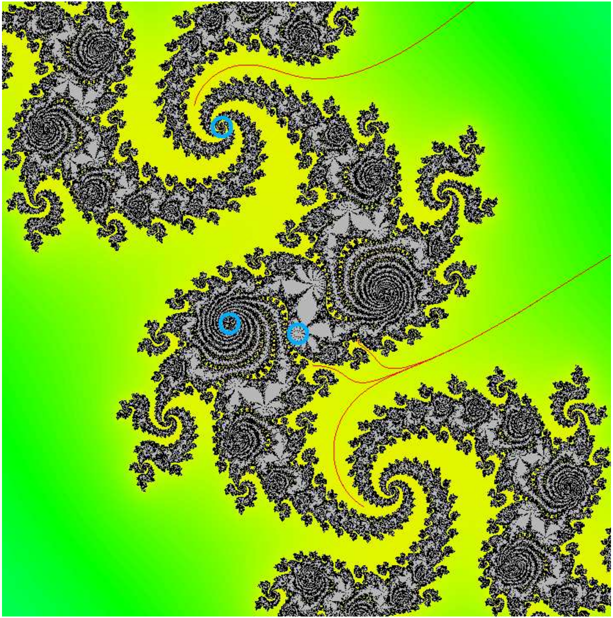
(Assuming only that $p_k \geq 3$ for all k, it follows that the differences $b_k - a_k \approx 2^{-p_1 \cdots p_k}$ tend faster than exponentially to zero as $k \to \infty$.)



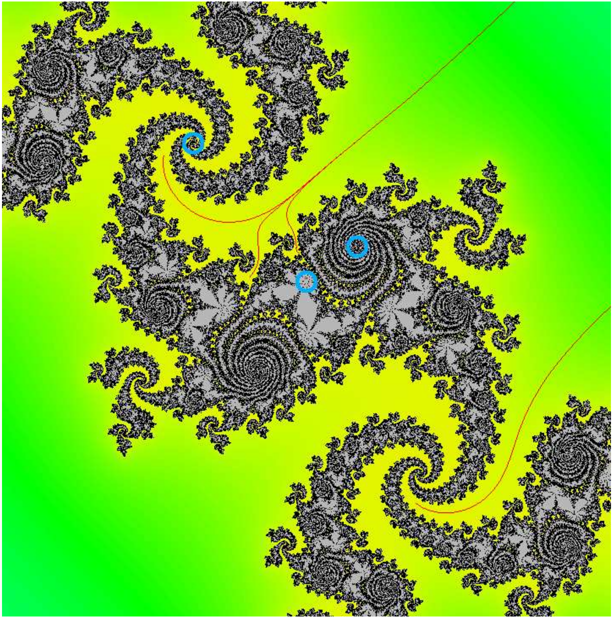
$H(1/3) \triangleright H(1/20) \triangleright 0$



$H(1/3) \triangleright H(-1/20) \triangleright 0$



$H(1/3) \triangleright H(1/7) \triangleright H(1/13) \triangleright 0$



$H(1/3) \triangleright H(-1/7) \triangleright H(1/13) \triangleright 0$

References

- A. Douady, Algorithms for computing angles in the Mandelbrot set, pp. 155–168 of “Chaotic Dynamics and Fractals”, ed. Barnsley & Demko, Acad. Press, 1986
- A. Douady, Chirurgie sur les applications holomorphes, Proc. Int. Cong. Math. Berkeley, A.M.S. 1987, 724–738.
- A. Douady and J. H. Hubbard, On the dynamics of polynomial-like mappings, Ann. Sci. Ec. Norm. Sup. (Paris) **18** (1985) 287–343.
- C. McMullen, “Complex Dynamics and Renormalization,” Princeton U. Press, 1994.
- J.Milnor, Self-similarity and hairiness in the Mandelbrot set, in “Computers in Geometry and Topology,” (M.C. Tangora, ed.), Marcel Dekker, New York, Basel, 1989, pp 211–257.

- J.Milnor, Local connectivity of Julia sets: expository lectures, pp. 67–116 of “The Mandelbrot set, Theme and Variations” edit: Tan Lei, LMS Lecture Note Series **274**, Cambridge U. Press, 2000.
- J.Milnor, Periodic orbits, external rays and the Mandelbrot set: An expository account, Asterisque **261** (2000) “Geometrie Complexe et Systemes Dynamiques”, pp. 277–333.
- J. Milnor, “Dynamics in One Complex Variable,” (3rd ed.) Princeton U. Press, 2006.
- D. E. K. Sørensen, Infinitely renormalizable quadratic polynomials, with non-locally connected Julia set, J. Geom. Anal. **10** (2000), 169–206