

*Points on the Circle:
from Pappus to Thurston*

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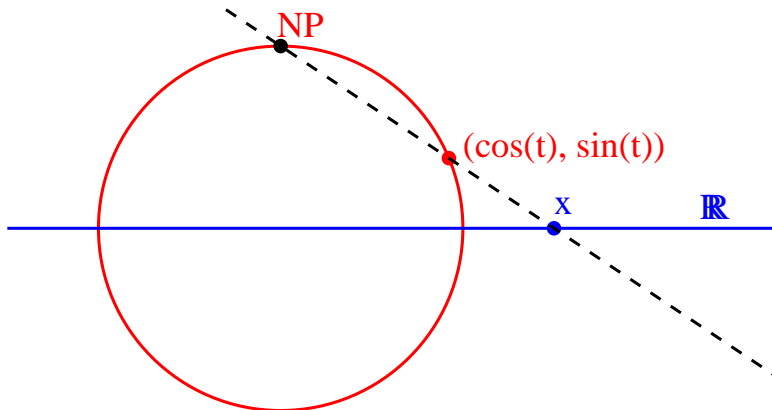
MAY 9, 2018

With help from

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The Circle: $\widehat{\mathbb{R}} \cong \mathbb{P}^1(\mathbb{R})$

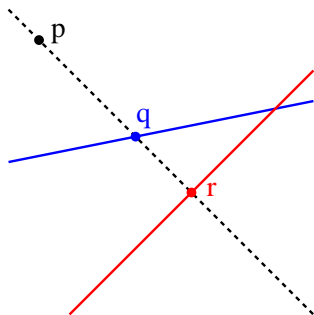
2.



Our “circle” will be the real projective line, $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$.

Projective (= fractional linear) transformations

3.

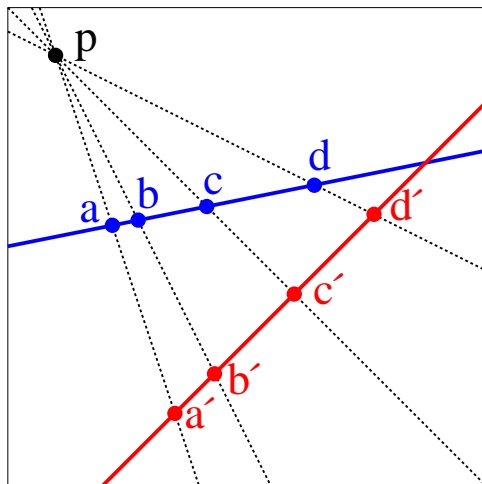


Geometrically: *Projecting from p , the point q on the blue line maps to the point r on the red line.*

Algebraically: A map from $\widehat{\mathbb{R}}$ to $\widehat{\mathbb{R}}$ is **fractional linear** if it has the form

$$x \mapsto \frac{ax + b}{cx + d} \quad \text{with} \quad ad - bc \neq 0 .$$

Essential Property: The action of the group of fractional linear transformations on $\widehat{\mathbb{R}}$ is **three point simply transitive**.



Pappus defined a numerical invariant, computed from the distances between four points on a line;
and proved that it is invariant under projective transformations.

Cross-Ratio: Four Points on the Projective Line.

5.

Definition (non-standard): For $a, b, c, d \in \mathbb{R}$, let

$$\begin{aligned}\mathbf{cr}(a, b, c, d) &= \mathbf{cr} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{(a-b)(c-d)}{(a-c)(b-d)} \in \widehat{\mathbb{R}}. \\ &= \frac{\text{product of row differences}}{\text{product of column differences}}\end{aligned}$$

Restriction: At least 3 of the 4 variables must be distinct.

There is a unique continuous extension to the case $a, b, c, d \in \widehat{\mathbb{R}}$. Then:

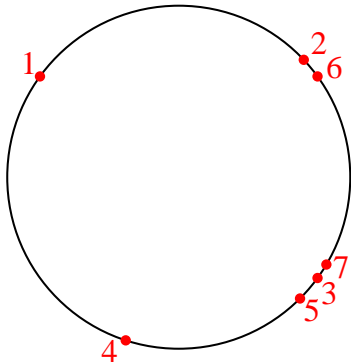
$$\mathbf{cr}(a, b, c, d) = 0 \iff a = b \text{ or } c = d,$$

$$\mathbf{cr}(a, b, c, d) = \infty \iff a = c \text{ or } b = d,$$

$$\mathbf{cr}(a, b, c, d) = 1 \iff a = d \text{ or } b = c,$$

$$\mathbf{cr}(a, b, c, d) \in \widehat{\mathbb{R}} \setminus \{0, 1, \infty\} \iff a, b, c, d \text{ all distinct.}$$

$$\mathbf{cr}(1, \infty, 0, x) = x \quad \text{for all } x.$$



Definition. The **moduli space** $\mathcal{M}_{0,n}(\mathbb{R}) = \mathcal{M}_n$ is the space of equivalence classes of ordered n -tuples (p_1, \dots, p_n) of distinct points of $\widehat{\mathbb{R}}$ modulo the action of the group of fractional linear transformations.

Thus (p_1, \dots, p_n) and (q_1, \dots, q_n) represent the same point of \mathcal{M}_n if and only if there is a fractional linear transformation \mathbf{g} such that

$$\mathbf{g}(p_j) = q_j \quad \text{for every } j .$$

Embedding \mathcal{M}_n into a product of many circles.

7.

Easy Lemma. *The n -tuples (p_1, \dots, p_n) and (q_1, \dots, q_n) represent the same point of \mathcal{M}_n if and only if:*

$$\mathbf{cr}(p_h, p_i, p_j, p_k) = \mathbf{cr}(q_h, q_i, q_j, q_k)$$

for every $1 \leq h < i < j < k \leq n$.

Thus we can embed \mathcal{M}_n into the $\binom{n}{4}$ -fold product of circles

$$\widehat{\mathbb{R}}^{\binom{n}{4}} = \prod_{0 \leq h < i < j < k \leq n} \widehat{\mathbb{R}},$$

sending the equivalence class of (p_1, \dots, p_n) into the $\binom{n}{4}$ -tuple of cross-ratios $\mathbf{cr}(p_h, p_i, p_j, p_k)$, where

$$1 \leq h < i < j < k \leq n.$$

A Non-Standard Definition of $\overline{\mathcal{M}}_n$

8.

Theorem (McDuff and Salamon). *The closure $\overline{\mathcal{M}}_n$ of \mathcal{M}_n within the torus*

$$\widehat{\mathbb{R}}^{\binom{n}{4}} = \prod_{0 \leq h < i < j < k \leq n} \widehat{\mathbb{R}}$$

is a smooth, compact, real-algebraic manifold of dimension $n - 3$.

Intuitive Proof that $\overline{\mathcal{M}}_n$ is a real-algebraic set.

Since the p_j are all distinct, we can put p_1, p_2, p_3 at $1, \infty, 0$, so that

$$\mathbf{cr}(p_1, p_2, p_3, p_k) = p_k \quad \text{for all } k .$$

Thus p_4, p_5, \dots, p_n are $n - 3$ independent variables, and determine all of the $\binom{n}{4}$ coordinate cross-ratios. Clearing denominators, we get a set of $\binom{n}{4} - 3$ defining polynomial equations.

The Simplest Cases $n = 3, 4$.

9.

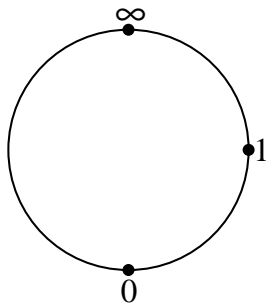
By definition, $\mathcal{M}_3 = \overline{\mathcal{M}}_3$ is a single point.

The subset $\mathcal{M}_4 \subset \mathbb{R}^{\binom{4}{4}} = \mathbb{R}$ is clearly just

$$\widehat{\mathbb{R}} \setminus \{0, 1, \infty\} = \mathbb{R} \setminus \{0, 1\};$$

and its closure within $\widehat{\mathbb{R}}$ is the entire circle: $\overline{\mathcal{M}}_4 \cong \widehat{\mathbb{R}}$.

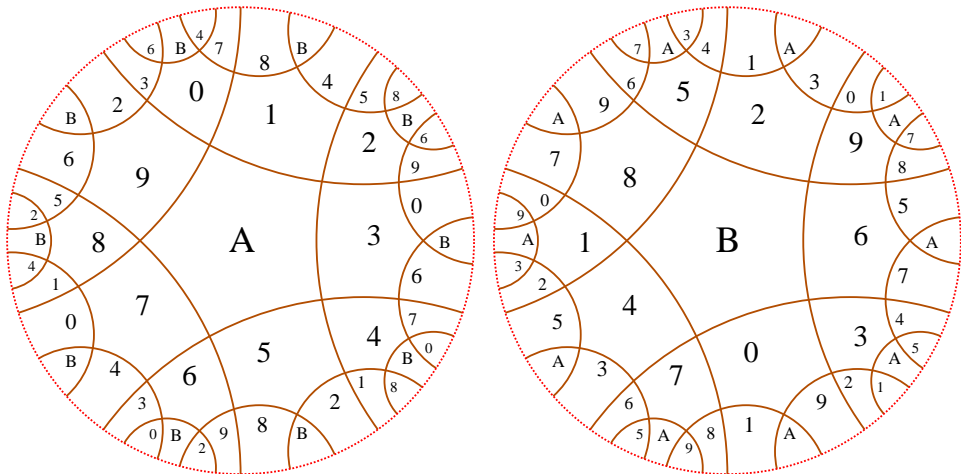
We should think of $\overline{\mathcal{M}}_4$ as a cell complex with three vertices and three edges:



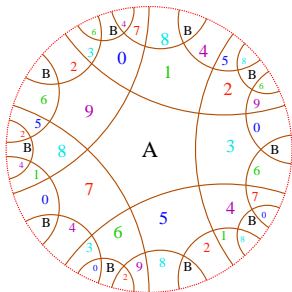
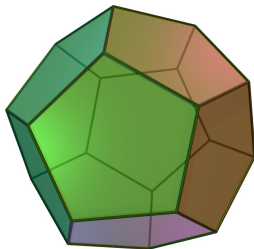
$\overline{\mathcal{M}}_5$ is a “hyperbolic dodecahedron”,

10.

covered by twelve *right angled hyperbolic pentagons*.



The interiors of the twelve pentagons
are the twelve connected components of \mathcal{M}_5 .



Both have isometry group of order 120:

$$\mathfrak{A}_5 \oplus (\mathbb{Z}/2);$$

$$\mathfrak{S}_5$$

In both cases, each face has an “opposite” face:

In the hyperbolic case, $A \leftrightarrow B$, $j \longleftrightarrow j + 5 \pmod{10}$.

Euler Characteristic:

$$\chi = 12 - 30 + 20 = 2, \quad \chi = 12 - 30 + 15 = -3.$$

Hyperbolic case $\implies \overline{\mathcal{M}}_5$ is non-orientable;
with no fixed point free involution.

Why Twelve Pentagons in $\overline{\mathcal{M}}_5$?

12.

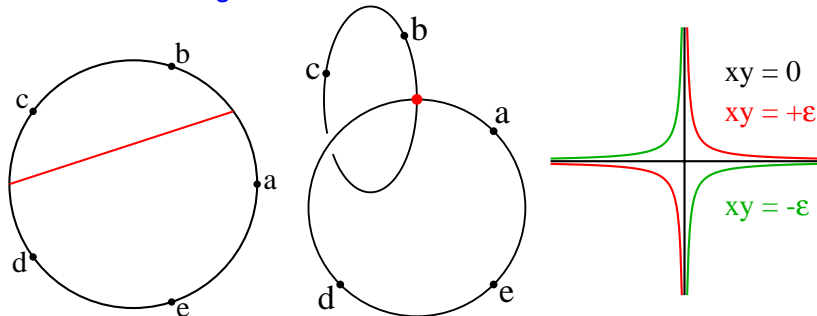
Each top dimensional cell in \mathcal{M}_n corresponds to one of the

$$\frac{(n-1)!}{2}$$

different ways of arranging the labels $1, 2, 3, \dots, n$ in cyclic order (up to orientation) around the circle.

Thus $\overline{\mathcal{M}}_5$ has $4!/2 = 12$ two-cells.

Within $\overline{\mathcal{M}}_5$ there are five different ways that two neighboring points can cross over each other to pass to a different face; hence five edges to each 2-cell.

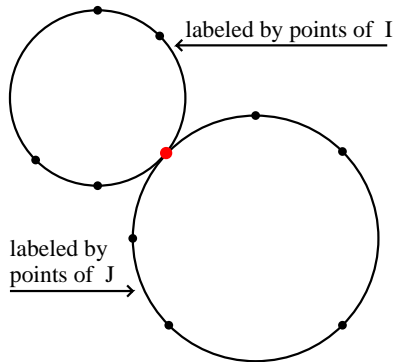


The Embedding $\varphi_{\mathbf{I}, \mathbf{J}} : \overline{\mathcal{M}}_{r+1} \times \overline{\mathcal{M}}_{s+1} \hookrightarrow \overline{\mathcal{M}}_n$. 13.

Let

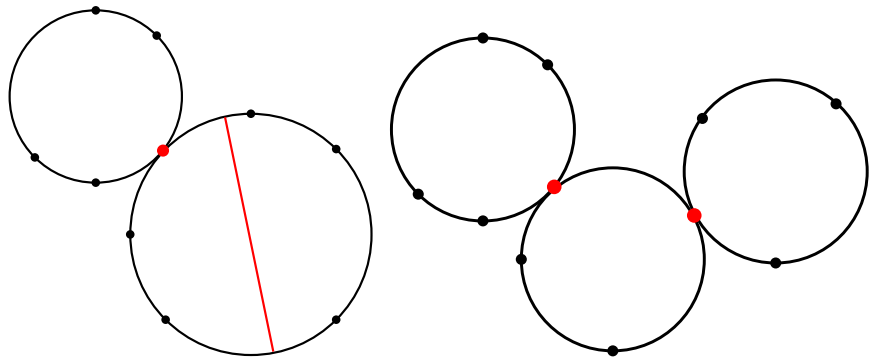
$$\{1, 2, \dots, n\} = \mathbf{I} \cup \mathbf{J}$$

be a partition into a set \mathbf{I} with $r \geq 2$ elements,
and a disjoint set \mathbf{J} with $s \geq 2$ elements, where $r + s = n$



The image of $\varphi_{\mathbf{I}, \mathbf{J}}$ is a union
of codimension one faces;

and every codimension
one face of $\overline{\mathcal{M}}_n$ is included
in the image of $\varphi_{\mathbf{I}, \mathbf{J}}$
for just one partition $\{\mathbf{I}, \mathbf{J}\}$
of $\{1, 2, \dots, n\}$.



Mumford Stability Condition:

Each circle must have at least three distinguished points.

For each $\mathbf{x} \in \overline{\mathcal{M}}_n$ and each list of distinct numbers h, i, j, k in $\{1, 2, \dots, n\}$, define the **limiting cross-ratio**

$$\mathbf{cr}_{h,i,j,k}(\mathbf{x}) \in \widehat{\mathbb{R}}$$

to be the limit, for any sequence of points $\mathbf{x}^\eta \in \mathcal{M}_n$ converging to \mathbf{x} , of the cross-ratios $\mathbf{cr}(p_h^\eta, p_i^\eta, p_j^\eta, p_k^\eta)$,

where each $(p_1^\eta, \dots, p_n^\eta) \in \widehat{\mathbb{R}}^n$ is a representative for the class $\mathbf{x}^\eta \in \mathcal{M}_n$.

Assertion. *The point $\mathbf{x} \in \overline{\mathcal{M}}_n$ belongs to the image,*

$$\mathbf{x} \in \varphi_{\mathbf{I},\mathbf{J}}(\overline{\mathcal{M}}_{r+1} \times \overline{\mathcal{M}}_{s+1}) \subset \overline{\mathcal{M}}_n,$$

if and only if

$$\mathbf{cr}_{i,i',j,j'}(\mathbf{x}) = 0$$

for every $i, i' \in \mathbf{I}$ and every $j, j' \in \mathbf{J}$.

There are $\binom{5}{2} = 10$ partitions of $\{1, 2, 3, 4, 5\}$ into subsets of order two and three. Hence there are ten embeddings

$$\overline{\mathcal{M}}_3 \times \overline{\mathcal{M}}_4 \cong \widehat{\mathbb{R}} \hookrightarrow \overline{\mathcal{M}}_5 .$$

These correspond to ten closed geodesics, each made up of three edges.

Thus there are $10 \times 3 = 30$ edges in $\overline{\mathcal{M}}_5$.

Each of these geodesics also contains three vertices, Here each vertex is counted twice since it belongs to two different geodesics, so there are $10 \times 3/2 = 15$ vertices.

Thus verifying that $\chi = 12 - 30 + 15 = -3$.

Example: $\overline{\mathcal{M}}_6$

17.

The are $\binom{6}{2} = 15$ partitions of $\{1, 2, 3, 4, 5, 6\}$ into subsets **I**, **J** of order two and four. Hence there are fifteen embeddings

$$\overline{\mathcal{M}}_3 \times \overline{\mathcal{M}}_5 \cong \overline{\mathcal{M}}_5 \hookrightarrow \overline{\mathcal{M}}_6 ;$$

where each copy of $\overline{\mathcal{M}}_5$ is made up of twelve pentagons.

Similarly there are $\binom{6}{3}/2 = 10$ partitions into two subsets of order three, yielding ten embeddings of the torus

$$\overline{\mathcal{M}}_4 \times \overline{\mathcal{M}}_4 \hookrightarrow \overline{\mathcal{M}}_6 .$$

Each copy of the torus is made up of $3 \times 3 = 9$ squares.

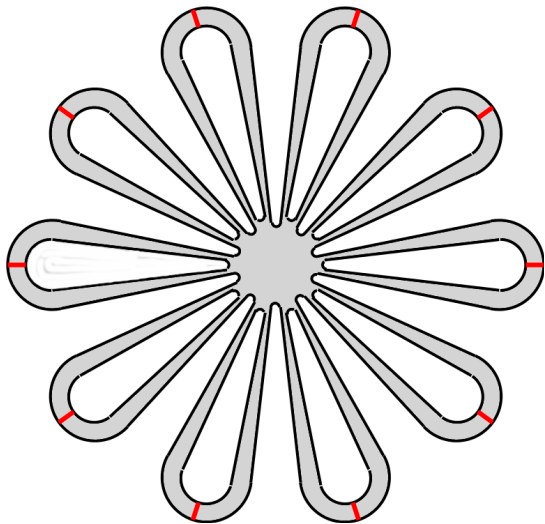
(Thus the 2-skeleton of $\overline{\mathcal{M}}_6$ consists of
 $15 \times 12 = 180$ pentagons, plus $10 \times 9 = 90$ squares.)

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According to Thurston, every smooth closed 3-manifold can be cut along embedded 2-spheres, tori, and/or Klein bottles into pieces, each of which has a locally homogeneous geometry.

Jaco-Shalen-Johannson Decomposition of $\overline{\mathcal{M}}_6$. 18.

Theorem. If we cut $\overline{\mathcal{M}}_6$ open along its ten embedded tori, then the remainder can be given the structure of a complete hyperbolic manifold of finite volume with twenty infinite cusps.

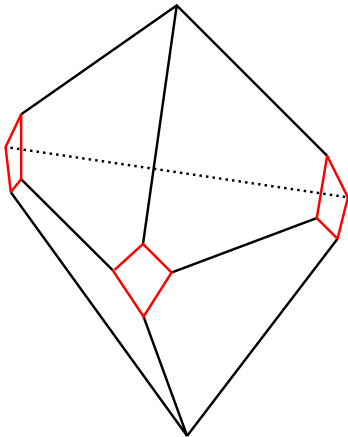
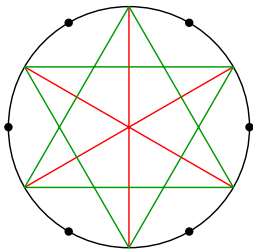


Corollary: The fundamental group $\pi_1(\overline{\mathcal{M}}_6)$ maps onto a free group on ten generators.

But $\pi_1(\overline{\mathcal{M}}_6)$ also contains free abelian groups $\mathbb{Z} \oplus \mathbb{Z}$.

Proof Outline.

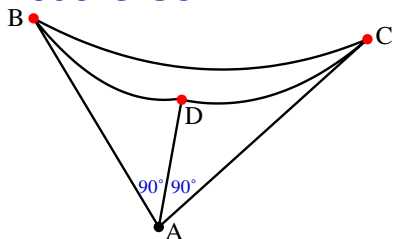
Each of the 60 3-cells in $\overline{\mathcal{M}}_6$ is bounded by 6 pentagons & 3 squares.



(Take the union of two tetrahedra with a face in common, and chop off three of the corners.)

We want 60 copies of this 3-cell to fit together to form a smooth manifold.

Thus we need all dihedral angles to be 90° !



In Hyperbolic 3-Space, choose three orthogonal lines of length ℓ starting at the point A. Then their convex closure is a tetrahedron with dihedral angle 90° along three of the edges.

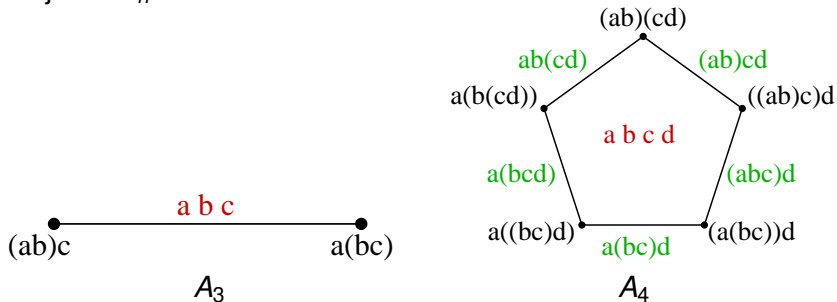
We need the dihedral angles along edges between B, C and D to be 45° . For ℓ finite, these angles are always $> 45^\circ$.

But as $\ell \rightarrow \infty$ these dihedral angles will tend to 45° .

Two copies yield a model 3-cell; but only by collapsing the three squares to points, and pushing them out to the sphere at infinity.

Thus $\overline{\mathcal{M}}_6$ with the 10 tori (or 90 squares) removed is a hyperbolic manifold tiled by 120 ideal tetrahedra.

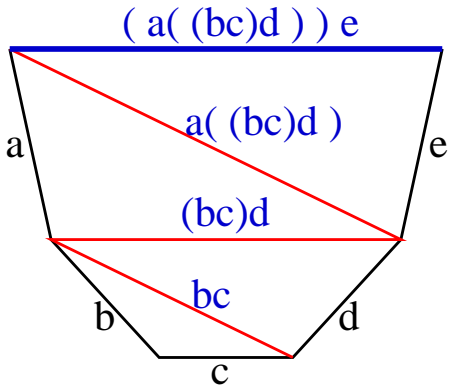
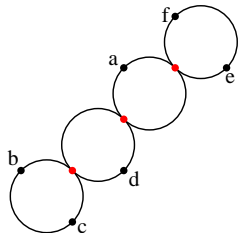
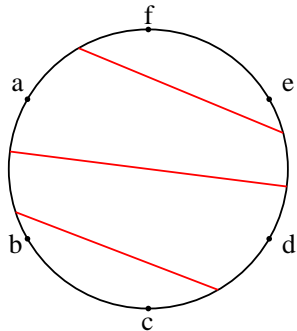
55 years ago Stasheff, while studying associativity for spaces with a continuous product operation, invented a sequence of objects A_n which we call **associahedra**.



The vertices of A_n correspond to the many ways of making sense of an n -fold non-associative product.

Theorem. *Each top-dimensional cell of $\overline{\mathcal{M}}_n$ is isomorphic as a cell complex to A_{n-1} .*

The top cells of $\overline{\mathcal{M}}_n$ are Associahedra: Proof Idea. 22.



Some References

23.

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