

Real Quadratic Rational Maps

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Advancing Bridges in Complex Dynamics

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Any real rational map f can be studied in two different ways:

As a piecewise monotone map from the circle $\mathbb{P}^1(\mathbb{R})$ to itself, f can be studied by very elementary methods.

But f always extends to a rational map $f_{\mathbb{C}}$ from the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ to itself, so that all of the tools of holomorphic dynamics are also available.

This talk will study the very special case of

Real Quadratic Maps, with Real Critical Points.

The Moduli Space $M = V/G$

2.

Let V be the smooth manifold consisting of all real quadratic maps

$$f(x) = \frac{ax^2 + bx + c}{a'x^2 + b'x + c'}$$

with real critical points. Let G be the group of orientation preserving fractional linear transformations

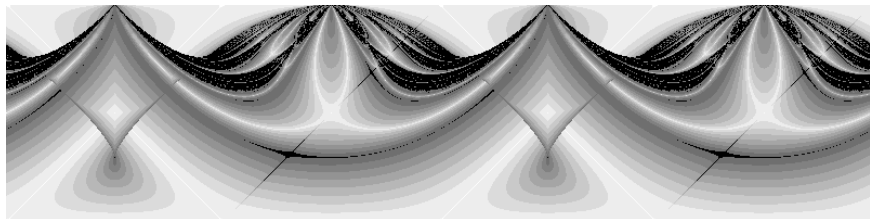
$$L(x) = \frac{\alpha x + \beta}{\gamma x + \delta} \quad \text{with} \quad \alpha\delta - \beta\gamma > 0,$$

acting on V by conjugation $f \mapsto L \circ f \circ L^{-1}$.

Then we can form the quotient space $M = V/G$.

Assertion: M is a Topological Cylinder

3.



This is a picture of part of the universal covering space of M .

The white regions correspond to maps such that both critical orbits converge to strongly attracting orbits of low period;

while the black points correspond to maps for which at least one critical orbit does not converge to an attracting periodic orbit.

Canonical Normal Form

4.

Main Lemma. Every map in V is conjugate under G to a unique map which satisfies three conditions:

(1): f has the form

$$f(x) = \frac{Ax^2 + B}{Cx^2 + D},$$

(Proof: Put the critical points at zero and infinity.)

(2): $AD - BC > 0$,

(Conjugate by $L(x) = -1/x$ if necessary.)

(3): $A^2 + C^2 = B^2 + D^2$.

(Conjugate by $L(x) = kx$. The required equation is satisfied for one and only one $k > 0$.)

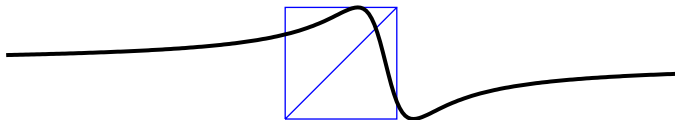
The resulting map f is then uniquely determined! □

(But A, B, C, D are only unique up to multiplication by a common non-zero constant.)

The Invariant Interval $f(\widehat{\mathbb{R}})$. 5.

If we think of f as a map from the circle $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ to itself, then:

The image $f(\widehat{\mathbb{R}})$ is a closed interval bounded by the two critical values.



Evidently all of the interesting dynamics is concentrated in this interval.

Theorem 1. *For any closed interval \mathcal{I} contained in the circle $\widehat{\mathbb{R}}$ there is one and only one map f in canonical form for which $f(\widehat{\mathbb{R}}) = \mathcal{I}$.*

Thus the moduli space M is homeomorphic to the set of all intervals $\mathcal{I} \subset \widehat{\mathbb{R}}$.

Proof: (Identifying $\widehat{\mathbb{R}}$ with \mathbb{R}/\mathbb{Z})

6.

For any map $f(x) = \frac{Ax^2 + B}{Cx^2 + D}$, the two critical values are

$$v_0 = f(0) = B/D \quad \text{and} \quad v_\infty = f(\infty) = A/C .$$

If we normalize the four coefficients A, B, C, D so that $A^2 + C^2 = B^2 + D^2 = 1$, then we can set

$$A = \sin(\pi t_\infty), \quad C = \cos(\pi t_\infty); \quad B = \sin(\pi t_0), \quad D = \cos(\pi t_0) .$$

Thus $v_\infty = \tan(\pi t_\infty)$ and $v_0 = \tan(\pi t_0)$.

By definition, t_∞ and t_0 are the **critical value angles**.

More generally, any angle $t \in \mathbb{R}/\mathbb{Z}$ corresponds to a unique point $\tan(\pi t) \in \widehat{\mathbb{R}}$.

Theorem 2. M is diffeomorphic to $\mathbb{R}/2\mathbb{Z} \times (0, 1)$. 7.

Proof. We have two distinct critical value angles

$$t_\infty, t_0 \in \mathbb{R}/\mathbb{Z},$$

Lift to points $\hat{t}_\infty, \hat{t}_0 \in \mathbb{R}$ so that $\hat{t}_0 < \hat{t}_\infty < \hat{t}_0 + 1$.

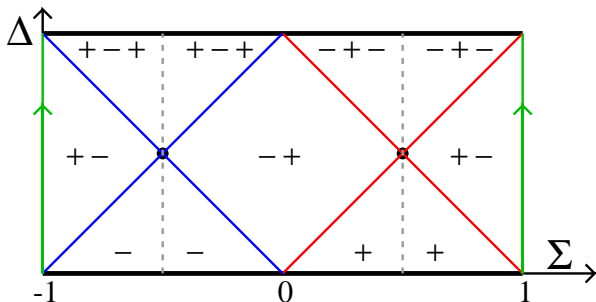
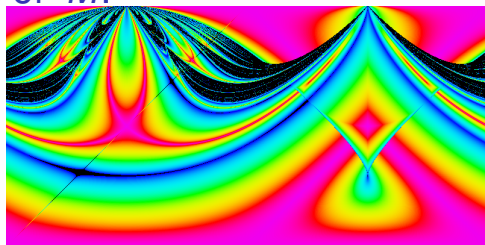
Then the corresponding point of M is uniquely determined by the two numbers

$$\Sigma = \hat{t}_\infty + \hat{t}_0 \pmod{2\mathbb{Z}}, \text{ and } \Delta = \hat{t}_\infty - \hat{t}_0 \in (0, 1). \quad \square$$

Here Δ is precisely the length of the interval $f(\hat{\mathbb{R}})$, lifted to \mathbb{R}/\mathbb{Z} .

Two Pictures of M .

8.

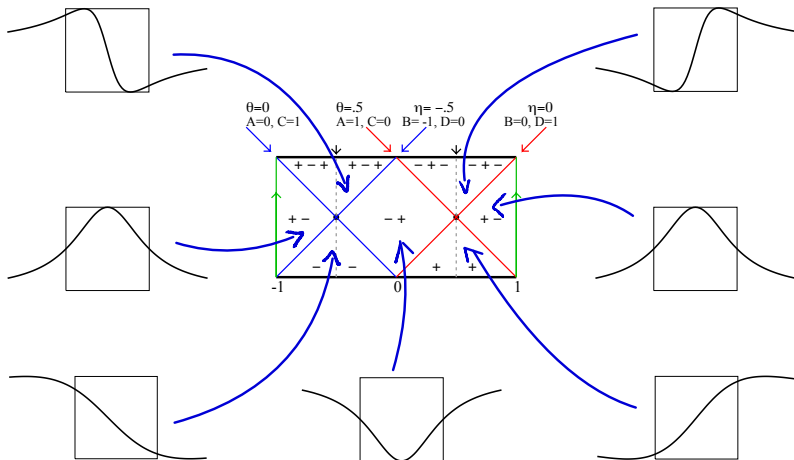


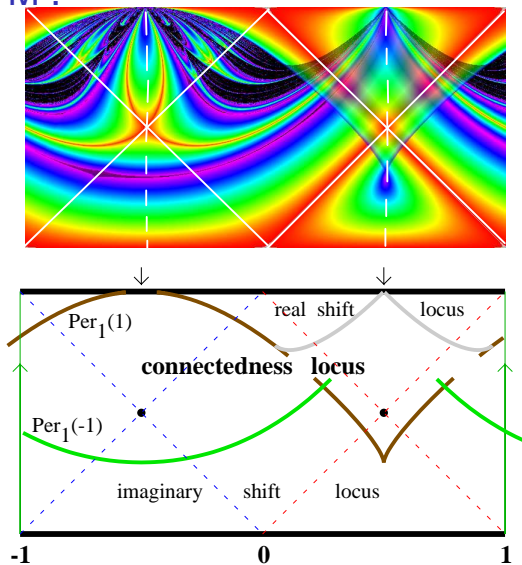
Red Lines: Polynomials (with critical fixed point).

Blue lines: "co-polynomials". (One critical point maps to the other.)

The Six Regions in Moduli Space

9.





Chebyshev curve: with $f(\text{critical value}) = \text{fixed point}$ with $\mu > 1$.

Every **complex** quadratic rational map either:

(1) belongs to the **connectedness locus**,

\Leftrightarrow connected Julia set, or

(2) belongs to the **shift locus**,

\Leftrightarrow totally disconnected Julia set.

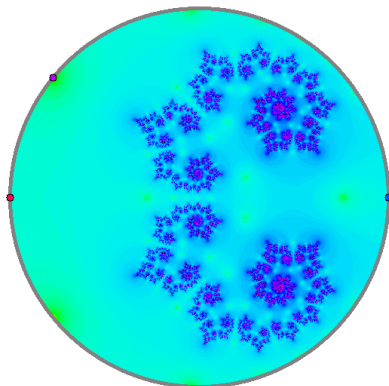
For our real maps f , there is a further distinction:

real shift locus

$$\Leftrightarrow J(f_{\mathbb{C}}) \subset \widehat{\mathbb{R}},$$

imaginary shift locus

$$\Leftrightarrow J(f_{\mathbb{C}}) \cap \widehat{\mathbb{R}} = \emptyset.$$



Critically finite maps fall into three types:

Hyperbolic.

*(A quadratic map is **hyperbolic** if both critical orbits converge to attracting periodic orbits.)*

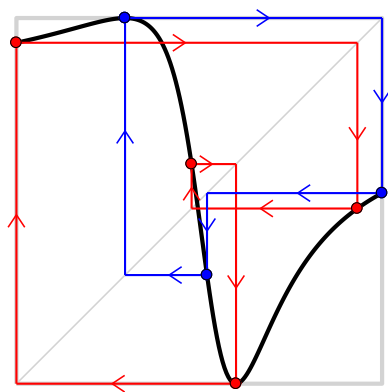
Half-Hyperbolic *if only one critical orbit converges to an attracting periodic orbit.*

Totally Non-Hyperbolic *if no critical orbit converges to an attracting periodic orbit.*

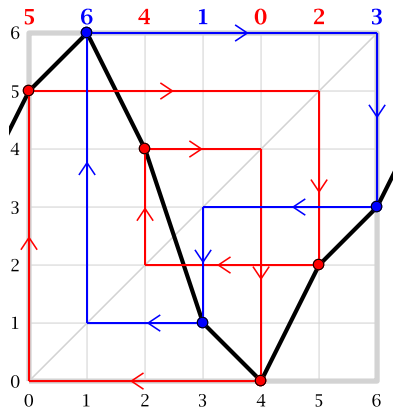
Every hyperbolic and critically finite f is the center point of a **hyperbolic component** in the connectedness locus.

But if f is totally non-hyperbolic then $J(f_{\mathbb{C}})$ is the entire Riemann sphere.

Combinatorics by Example: The Wittner Map 13.



With combinatorics:



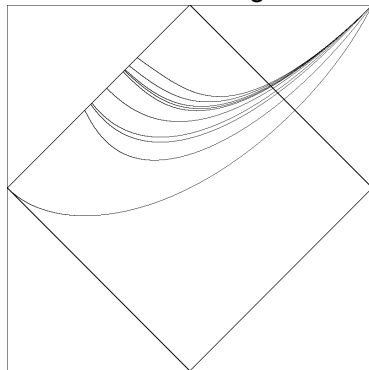
(5, 6, 4, 1, 0, 2, 3).

Theorem: Any critically finite point in M is uniquely determined by its combinatorics.

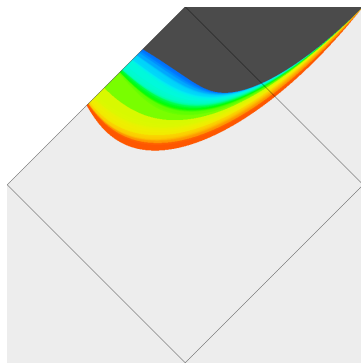
[movie]

By definition, a **bone** in M is a maximal smooth curve on which one of the two critical points is periodic.

In the unimodal region:



Theorem of Filom + Yan Gao:
Each bone in the unimodal region is a smooth curve from polynomial to co-polynomial;

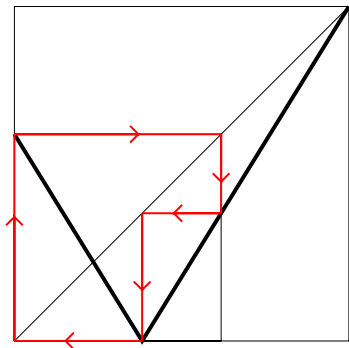


and each locus of constant topological entropy is connected.

Maps with Constant |Slope|.

15.

Misiurewicz showed that a map with constant $|\text{slope}| = s > 1$ has topological entropy $h = \log(s)$.



Here $s = (\sqrt{5} + 1)/2$.

Theorem. To every critically finite co-polynomial there corresponds a critically finite polynomial.

(Conversely to **almost** every critically finite polynomial there corresponds a critically finite co-polynomial.)

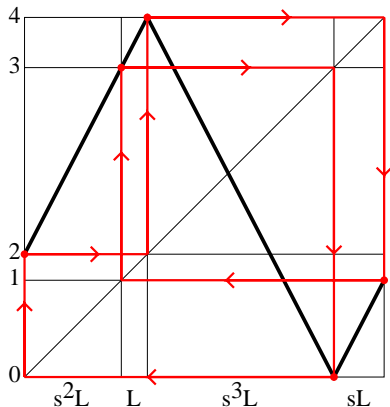
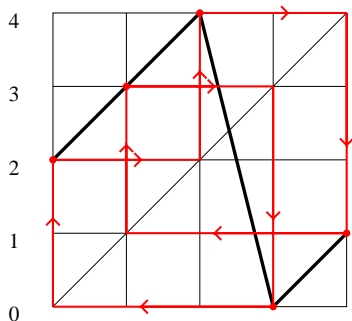
The Filom-Pilgrim family of maps.

16.

Given $0 < p/q < 1$ consider the combinatorics

$(m_0, m_1, \dots, m_{q-1})$ with $m_k \equiv k + p \pmod{q}$.

Here is the PL model
for $p/q = 2/5$.



Since the longest edge maps onto the entire interval, we get

$$s^4 - s^3 - s^2 - s - 1 = 0.$$

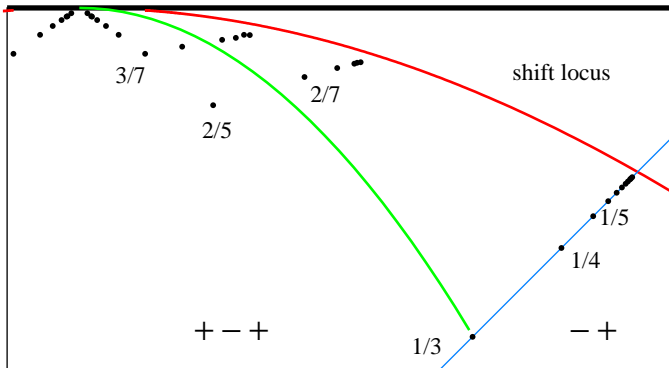
Thus the topological entropy $\log(s_q)$ depends only on q .

Theorem (Filom and Pilgrim).

17.

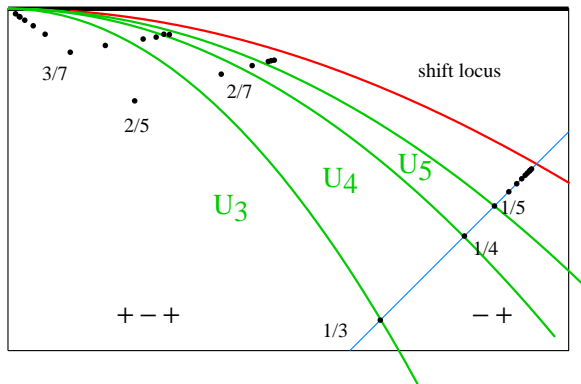
In the $+ - +$ region, loci of constant topological entropy can have arbitrarily many connected components.

Step 1: Each hyperbolic component $H(p/q)$ contains a curve leading from the center point to the ideal point $(\Sigma, \Delta) = (-.5, 1)$.



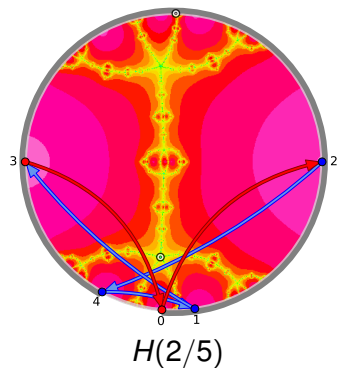
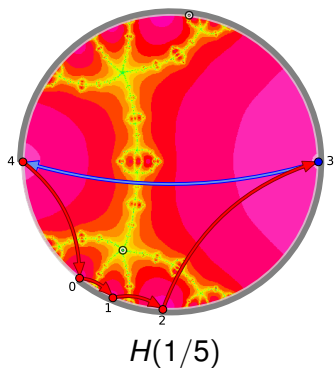
Furthermore these curves depend monotonically on p/q .

For each $q \geq 3$ there is a curve C_q of constant entropy $\log(s_q)$ which extends from the left ideal point $(-.5, 1)$ through $f_{1/q}$ to the right and ideal point $(+.5, 1)$. (Compare Slide 14.)



The curves C_q divide the square region $-0.5 \leq \Sigma \leq 0.5$ into disjoint connected open sets U_3, U_4, U_5, \dots .

If $q > n(n-1)$ is prime, there is a p so that $f_{p/q} \in U_n$.



Key Lemma. As f tends to the ideal point $(-0.5, 1)$ within the hyperbolic component $H(p/q)$, the multiplier of the fixed point of f_c in the upper half plane tends to $e^{2\pi i p/q}$.

If we are given two of the fixed point multipliers for a quadratic rational map, then the third is given by

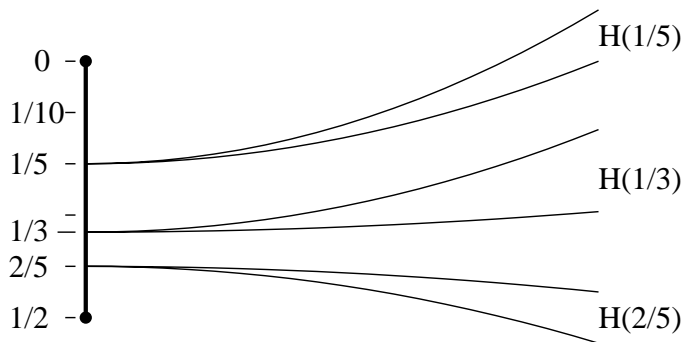
$$\mu_3 = \frac{2 - \mu_1 - \mu_2}{1 - \mu_1\mu_2}.$$

Now suppose that $\mu_1 = r e^{i\theta}$ and $\mu_2 = r e^{-i\theta}$. Then

$$\mu_3 = \frac{2 - 2r \cos(\theta)}{1 - r^2}.$$

If the map has real critical points then we must have $r \geq 1$.

If $\mu_3 \rightarrow -\infty$, it follows easily that $r \rightarrow 1$. \square



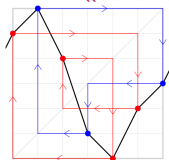
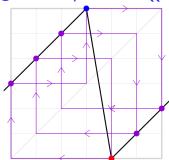
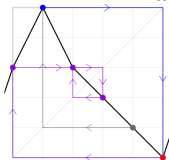
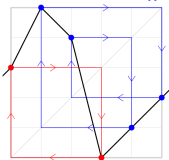
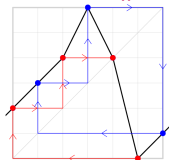
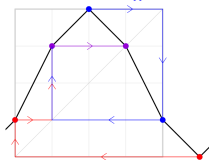
This is a hypothetical picture of what we would get if we replace the ideal point $(-.5, 1)$ by an entire vertical interval of angles $0 \leq \theta \leq 1/2$.





This completes the outlined proof of non-monotonicity.

For further details see Filom and Pilgrim's paper.

Examples of the Thurston Pullback Map.

22.

Wittner $((5, 6, 1, 0, 2, 3))$ Filom-Pilgrim 3/7 : $((3, 4, 5, 6, 0, 1, 2))$ Weakly Obstructed $((3, 4, 3, 2, 1, 0))$ Str. Obstructed $((3, 5, 4, 0, 1, 2))$ Str. Obstructed $((2, 3, 4, 6, 4, 0, 1))$ Exceptional $((1, 3, 4, 3, 1, 0))$ 

-  **Milnor, Geometry and Dynamics of Quadratic Rational Maps,** WITH AN APPENDIX BY THE AUTHOR AND TAN LEI. **Experiment. Math.** 2 (1) 37 - 83, 1993. (Also in: “Collected Papers of John Milnor VI: Dynamical Systems (1953-2000), **Amer. Math. Soc.** (2012) 371 – 438 ”.)
-  **Filom and Pilgrim, On the Non-Monotonicity of Entropy for a Class of Real Quadratic Rational Maps.** **Journal of Modern Dynamics** 16, 225–254, 2020.
-  **Yan Gao, Monotonicity of entropy for unimodal real quadratic rational maps.** [arXiv:2009.03797](https://arxiv.org/abs/2009.03797)
-  **Bonifant, Milnor and Sutherland, The W. Thurston Algorithm for Real Quadratic Rational Maps** [arXiv:2009.10147](https://arxiv.org/abs/2009.10147) [math.DS]
In preparation:
A more detailed manuscript, plus an interactive web site.