Curvature and Résidu Itératif

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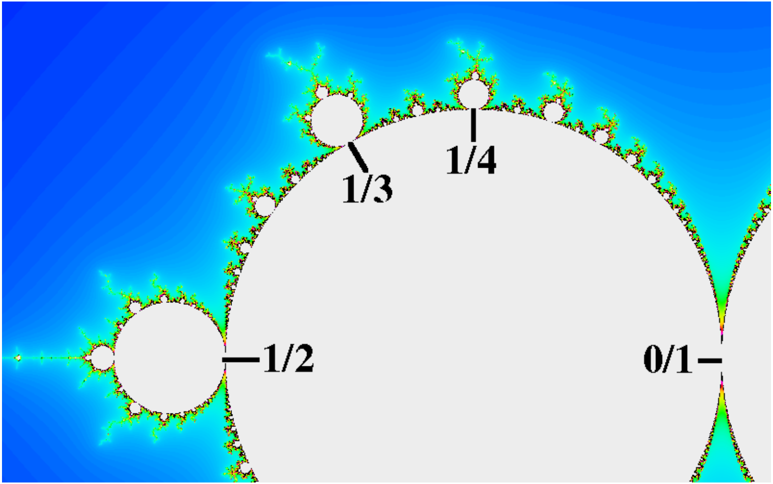
Example: The Rounded Mandelbrot Set



Connectedness locus for the family of maps

$$g_\lambda(z) = z^2 + \lambda\, z\ .$$

Two Fixed Point Invariants.

Consider an isolated fixed point $z_0 = f(z_0)$ of a holomorphic map $f : \mathbb{C} \to \mathbb{C}$.

One basic invariant is the ***multiplier*** $\lambda = f'(z_0)$.

Another is the ***holomorphic index***

$$\mathbf{ind}(f, z_0) = \frac{1}{2\pi i} \oint_{z_0} \frac{dz}{z - f(z)} .$$

For a fixed point with $\lambda \neq 1$, it is not hard to check that

$$\mathbf{ind}(f, z_0) = \frac{1}{1 - \lambda}.$$

If $\lambda = 1$, then for any small $\epsilon \neq 0$, the one local fixed point for f will split into n nearby fixed points $z_1, \ldots, z_n$ for $f + \epsilon$,

where $n \geq 2$ is called the ***fixed point multiplicity***.

Furthermore: $\lambda_j = (f + \epsilon)'(z_j) \neq 1$.

Assertion : $\mathbf{ind}(f, z_0) = \lim_{\epsilon \to 0} \sum_{j=1}^{n} \mathbf{ind}(f + \epsilon, z_j) = \lim_{\epsilon \to 0} \sum_{j=1}^{n} \frac{1}{1 - \lambda_j} .$

Résidu Itératif (Jean Écalle, 1976).

Definition.
If $\lambda = 1$, the difference

$$\textbf{résit}(f,\, z_0) \;=\; \frac{n}{2} - \textbf{ind}(f,\, z_0)$$

is called the **résidu itératif**.

Theorem. For any integer $k \geq 1$:

$$\textbf{résit}(f^{\circ k},\, z_0) \;=\; \frac{1}{k}\,\textbf{résit}(f,\, z_0)\;. \qquad (1)$$

Proof. For $\epsilon \approx 0$ the fixed point with multiplier one for f splits into n fixed points for $f + \epsilon$ with multipliers $\lambda_1, \ldots, \lambda_n \approx 1$.

Therefore $\quad \textbf{résit}(f^{\circ k},\, z_0) = \lim_{\epsilon \to 0} \sum_{j=1}^{n} \left(\frac{1}{2} - \frac{1}{1 - \lambda_j^k} \right)$.

Lemma. $\quad \left(\frac{1}{2} - \frac{1}{1-\lambda^k}\right) \;=\; \frac{1}{k}\left(\frac{1}{2} - \frac{1}{1-\lambda}\right) \;+\; o(1) \quad \text{as} \quad \lambda \to 1$.

Equation (1) then follows easily. $\square$

Extended definition (Buff and Epstein, 2002).

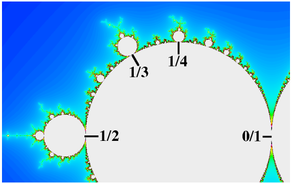
The résidu itératif can be defined at ***any*** parabolic fixed point, so that $\textbf{résit}(f^{\circ k}, z_0) = \textbf{résit}(f, z_0)/k$.

If $\lambda_0 = f'(z_0)$ is a p-th root of unity, simply define:

$$\textbf{résit}(f, z_0) := p \cdot \textbf{résit}(f^{\circ p}, z_0) ,$$

using the Ecalle definition on the right.

We want to relate the résidu itératif to curvature in parameter space.



In the family $\{z \mapsto z^2 + \lambda z\}$, each root of unity $\lambda_0 = e^{2\pi i\, q/p}$ is a common boundary point for the main hyperbolic component H, and for a satellite component $S(q/p)$.

Theorem. *The real part* $\Re(\textbf{résit}(g_{\lambda_0}, 0))$ *is equal to the average of the two curvatures:*

$$K(\partial H, \lambda_0) = +1 \quad \text{and} \quad K(\partial S(q/p), \lambda_0) .$$

Examples:

q/p	**résit**$(g_{\exp(2\pi i\, q/p)})$	K_S	K_S/p^2
0/1	1	1	1
1/6	18.283 + 1.182 i	35.585	.988
1/5	13.065 + .677 i	25.130	1.005
1/4	8.748 + .316 i	16.497	1.031
1/3	5.320 + .094 i	9.639	1.071
2/5	12.962 - .058 i	24.924	.997
1/2	2.75	4.5	1.125

Here

$$\Re(\textbf{résit}) = (1 + K_s)/2 \quad \Longleftrightarrow \quad K_S = 2\,\Re(\textbf{résit}) - 1\ .$$

A Convenient Notation.

Let $\alpha \mapsto \beta$ be a twice differentiable (or holomorphic) map with $d\beta/d\alpha \neq 0$.

Define the ***nonlinearity*** of $\alpha \mapsto \beta$ to be the ratio

$$(\!(\alpha,\, \beta)\!) \;=\; \frac{d^2\beta/d\alpha^2}{d\beta/d\alpha}\,.$$

Thus $\quad (\!(\alpha,\, \beta)\!) = 0 \quad \Longleftrightarrow \quad \beta = c_1\alpha + c_2\,.$

The Chain Rule for $\alpha \mapsto \beta \mapsto \gamma$:

$$(\!(\alpha,\, \gamma)\!) \;=\; (\!(\alpha,\, \beta)\!) + (\!(\beta,\, \gamma)\!)\frac{d\beta}{d\alpha}\,.$$

This follows from the identity

$$\log\frac{d\gamma}{d\alpha} \;\equiv\; \log\frac{d\beta}{d\alpha} + \log\frac{d\gamma}{d\beta} \qquad \pmod{2\pi i}\,,$$

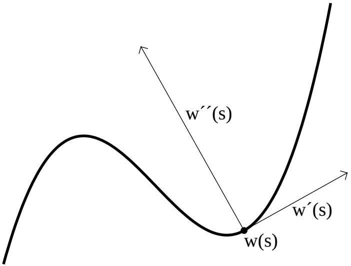
by differentiating with respect to α.

A Simple Example.

The chain rule for the composition $c\alpha \mapsto \alpha \mapsto \beta$ yields

$$\begin{aligned} (\!(c\alpha,\, \beta)\!) &= (\!(c\alpha,\, \alpha)\!) + (\!(\alpha,\, \beta)\!)\frac{d\,\alpha}{d\,c\alpha} \\ &= 0 + (\!(\alpha,\, \beta)\!)/c\,. \end{aligned}$$

Curvature.



For a curve $s \mapsto w(s)$ parametrized by arclength,
we have $|w'| = |dw/ds| = 1$, and

$$((s, w)) = w''/w' = iK, \quad \text{hence} \quad K = \Im((s, w)) .$$

For an arbitrary smooth parametrization $t \mapsto s \mapsto w$,
it follows that $((t, w)) = ((t, s)) + i K\, ds/dt$, hence

$$\Im((t, w)) = 0 + K \frac{ds}{dt} = K \left| \frac{dw}{dt} \right| .$$

Again let $g_\lambda(z) = z^2 + \lambda z$.

Thus g_λ has a fixed point at $z = 0$ with multiplier λ.
If $\lambda \approx \lambda_0 = e^{2\pi i q/p}$, then g_λ has a period p orbit near zero.
Let μ be its multiplier. Then $\textbf{ind}(g_\lambda^{\circ p}, 0) = \left(\frac{1}{1-\lambda^p} + \frac{p}{1-\mu}\right)$.

$$\Longrightarrow \qquad \textbf{ind}(g_{\lambda_0}^{\circ p}, 0) = \lim_{\lambda \to \lambda_0} \left(\frac{1}{1-\lambda^p} + \frac{p}{1-\mu}\right).$$

Corollaries:

1. $\mu = 1$ if and only if $\lambda^p = 1$.

2. μ is locally a holomorphic function of λ, or of λ^p.

3. The derivative at λ_0 is $d\mu/d\lambda^p = -p$,

$$\Longleftrightarrow \quad d\log\mu/d\log\lambda = -p^2 .$$

4. $\textbf{ind}(f_{\lambda_0}^{\circ p}) = ((1-\lambda^p,\ 1-\mu))/2$ evaluated at λ_0,

$$= -((\lambda^p,\ \mu))/2 .$$

Computation of the résidu itératif.

Theorem: For any $k \geq 1$ we have

$$\textbf{résit}(f_{\lambda_0}^{\circ k}, 0) \;=\; \frac{((\log\lambda,\ \log\mu))}{2\,k}\ .$$

Proof outline: Start with $-\textbf{ind}(f_{\lambda_0}^{\circ p},\ 0) = ((\lambda^p,\ \mu))/2$.

First express $((\lambda^p,\ \mu))$ as a linear function of $((\log\lambda,\ \mu))$, using the chain rule for the composition $\log\lambda^p \mapsto \lambda^p \mapsto \mu$ $\big($where $\log(\lambda^p) = p\log(\lambda)\big)$.

Then express $((\log\lambda,\ \mu))$ as a function of $((\log\lambda,\ \log\mu))$, using the chain rule for the composition $\log\lambda \mapsto \log\mu \mapsto \mu$. The result will be

$$-\textbf{ind}(f_{\lambda_0}^{\circ p},\ 0) \;=\; \frac{((\log\lambda,\ \log\mu))}{2p} - \frac{p+1}{2}\ .$$

Adding $\frac{p+1}{2}$ to both sides, we obtain

$$\textbf{résit}(f_{\lambda_0}^{\circ p}, 0) \;=\; \frac{((\log\lambda,\ \log\mu))}{2\,p}\ . \quad \square$$

For a holomorphically parametrized family of maps

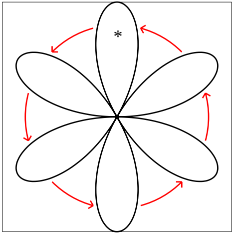
$$F_\xi : \mathbb{C} \to \mathbb{C}\,.$$

Suppose that:

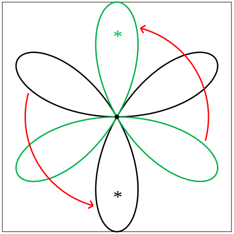
(1) each F_ξ has a specified fixed point $z_0(\xi)$ which varies holomorphically with ξ,

(2) the multiplier $\lambda = \lambda(\xi)$ of this fixed point satisfies $d\lambda/d\xi \neq 0$, and

(3) ξ_0 is a parameter for which $z_0(\xi_0)$ is a fixed point of ***parabolic multiplicity*** $m = 1$.

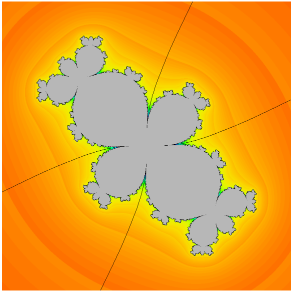


parabolic multiplicity $m = 1$

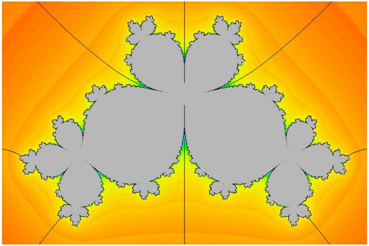


parabolic multiplicity $m = 2$

Cubic Examples



$f(z) = z^3 + i\,z$, parabolic multiplicity $m = 1$



$z \mapsto z^3 + i\,z^2 - z$, parabolic multiplicity $m = 2$

Recall the conditions for a family of maps

$$F_\xi : \mathbb{C} \to \mathbb{C}\ .$$

Suppose that:

(1) each F_ξ has a specified fixed point $z_0(\xi)$ which varies holomorphically with ξ,

(2) the multiplier $\lambda = \lambda(\xi)$ of this fixed point satisfies $d\lambda/d\xi \neq 0$, and

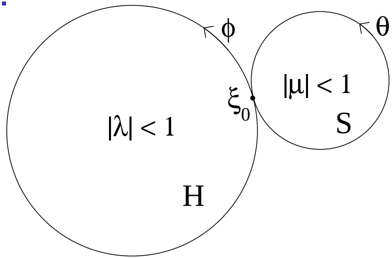
(3) ξ_0 is a parameter for which $z_0(\xi_0)$ is a fixed point of ***parabolic multiplicity one***.

Theorem. *Then*

$$\textit{\textbf{résit}}(F_{\xi_0},\, z_0) \;=\; \frac{(\!(\log\lambda,\, \log\mu)\!)}{2}$$

$$=\; \frac{(\!(\log\lambda,\, \xi)\!) + p^2(\!(\log\mu,\, \xi)\!)}{2}\ .$$

Curvature Again.



Make the substitutions $\lambda = e^{i\phi}$ and $\mu = e^{i\theta}$.
Thus real values of ϕ (or θ) parametrize ∂H (or ∂S).
Then

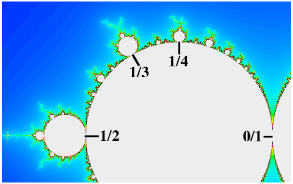
$$\textbf{résit}(F_{\xi_0}) = \frac{(\!(\log\lambda,\,\xi)\!) + p^2(\!(\log\mu,\,\xi)\!)}{2} = \frac{(\!(\phi,\,\xi)\!) + p^2(\!(\theta,\,\xi)\!)}{2\,i}$$

Corollary.

$$\Re\big(\textbf{résit}(F_{\xi_0})\big) = \frac{K(\partial H,\,\xi_0) + K(\partial S,\,\xi_0)}{2}\left|\frac{d\xi}{d\lambda}\right| .$$

Limiting Shape?

What can one say about the “sizes” and “shapes” of the various satellites $S(q/p)$ of the rounded Mandelbrot set?



Question: Given a sequence of fractions q_j/p_j tending to a limit, when do the $S(q_j/p_j)$ have a limiting shape?

Each $S(q/p)$ has a prefered ***center point*** $c = c(q/p)$, defined by the equation $\mu = 0$.

*Define the **"radius"** $r = r(q/p)$ to be the distance $|c - \lambda_0|$, where $\lambda_0 = e^{2\pi i\, q/p}$ is the root point.*

Then the product $r\, K_S$ associated with a given satellite is scale invariant measure of distortion,

equal to one for a round disk.

Approximating $1/3$ by Farey Neighbors

From the left			From the right		
q/p	$2\,\textbf{résit}/p^2$	$r_S K_S$	q/p	$2\,\textbf{résit}/p^2$	$r_S K_S$
1/4	1.094 +.039 i	1.014	1/2	1.375	1.062
3/10	.944 -.017 i	.965	3/8	.963 + .015 i	.973
5/16	.926 -.047 i	.959	5/14	.927 + .046 i	.958
7/22	.926 -.063 i	.959	7/20	.924 + .063 i	.957
9/28	.930 -.072 i	.960	9/26	.927 + .072 i	.959
⋮	⋮	⋮	⋮	⋮	⋮
100/301	.964 -.082 i	.978	100/299	.962 + .081 i	.977
370/1111	.967 -.081 i	.980	370/1109	.966+ .080 i	.980
550/1651	.968 -.081 i	.981	550/1649	.966+ .079 i	.979
1000/3001	.968 -.081 i	.980	1000/2999	.966+ .079 i	.979
3700/11101	.968 -.080 i	.981	3700/11099	.967+.079 i	.980
9100/27301	.970 -.081 i	.980	9100/27299	.968+.080 i	.981

Approximating $(\sqrt{5}-1)/2$.

(Illustrating an ongoing project by
D. Dudko, M. Lyubich and N. Selinger.)

From the left			From the right		
q/p	$2\,\textbf{résit}/p^2$	$r_S K_S$	q/p	$2\,\textbf{résit}/p^2$	$r_S K_S$
1/2	1.375	1.062	2/3	1.182-.021 i	1.034
3/5	1.037+.005 i	.997	5/8	.963-.016 i	.973
8/13	.921+.009i	.956	13/21	.898-.013 i	.946
21/34	.886 +.011 i	.944	34/55	.879 -.012 i	.937
55/89	.876 +.011 i	.935	89/144	.874 -.012 i	.935
144/233	.873 +.011 i	.934	233/377	.872-.012 i	.933
377/610	.872 -.012 i	.933	610/987	.872 -.012 i	.933
987/1597	.872 +.012 i	.933	1597/2584	.872 -.012 i	.933
2584/4181	.872 +.011 i	.933	4181/6765	.872-.012 i	.933
6765/10946	.872 +.012 i	.933	10946/17711	.872 -.012 i	.933