

Two Moduli Spaces

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Outline: Two Examples.

2.

The object of this talk will be to describe two examples of smooth group actions on smooth manifolds.

Easier Example (Divisors on \mathbb{P}^1):

The group $G(\mathbb{P}^1) = \mathrm{PGL}_2(\mathbb{C})$ of Möbius automorphisms of the Riemann sphere \mathbb{P}^1 acts on the space \mathcal{D}_n of effective divisors of degree n on \mathbb{P}^1 , with quotient space $\mathcal{D}_n/G(\mathbb{P}^1)$.

Much Harder Example (Curves in \mathbb{P}^2):

The group $G(\mathbb{P}^2) = \mathrm{PGL}_3(\mathbb{C})$ of projective automorphisms of the complex projective plane \mathbb{P}^2 , acts on the projective compactification \mathcal{C}_n of the space of algebraic curves of degree n in \mathbb{P}^2 , with quotient space $\mathcal{C}_n/G(\mathbb{P}^2)$.

In both cases, some parts of the quotient space are beautiful objects to study, but other parts are rather nasty.

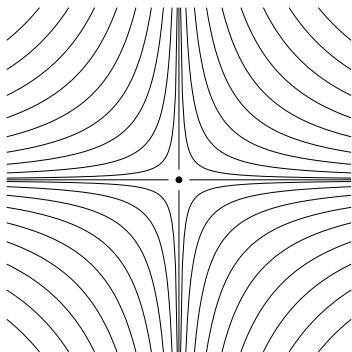
Basic Problem: Which parts are which?

A Toy Example

3.

The additive group G of real numbers acts on \mathbb{R}^2 by $\mathbf{g}_t(x, y) = (e^t x, e^{-t} y)$.

Most orbits are smooth curves; but the origin is a single point orbit.



If we remove the origin, then the quotient space

$$\left(\mathbb{R}^2 \setminus \{(0, 0)\}\right) / G$$

is locally a smooth manifold.

But it is only locally Hausdorff.

Part 1. The Space \mathcal{D}_n of Degree n Divisors on \mathbb{P}^1 . 4.

Definition: An **effective divisor** \mathcal{D} of degree n on the Riemann sphere $\mathbb{P}^1 = \mathbb{P}^1(\mathbb{C})$ is a formal sum

$$\mathcal{D} = m_1 \langle \mathbf{p}_1 \rangle + \cdots + m_k \langle \mathbf{p}_k \rangle ,$$

where the $m_j > 0$ are integers with $\sum_j m_j = n$, and the \mathbf{p}_j are distinct points of \mathbb{P}^1 .

Each such \mathcal{D} can be identified with the set of zeros, counted with multiplicity, for some non-zero homogeneous polynomial

$$\Phi(x, y) = c_0 x^n + c_1 x^{n-1} y + \cdots + c_n y^n .$$

It follows that the space \mathcal{D}_n of all such divisors is isomorphic to the projective space $\mathbb{P}^n(\mathbb{C})$.

The group $G = G(\mathbb{P}^1)$ of Möbius automorphisms of \mathbb{P}^1 acts on \mathcal{D}_n .

Two integer invariants under the action of G :

- *The number of points $k = \#\mathcal{D}$ in the **support***

$$|\mathcal{D}| = \{\mathbf{p}_1, \dots, \mathbf{p}_k\} \subset \mathbb{P}^1 .$$

- *The maximum $m_{\max} = \max\{m_1, \dots, m_k\}$ of the multiplicities of the various points of $|\mathcal{D}|$.*

Finite Stabilizers

Definition. The **stabilizer** $G_{\mathcal{D}}$ of a divisor \mathcal{D} is the subgroup of G consisting of all $\mathbf{g} \in G$ with $\mathbf{g}(\mathcal{D}) = \mathcal{D}$.

Lemma. *The stabilizer $G_{\mathcal{D}}$ is finite if and only if the support $|\mathcal{D}| \subset \mathbb{P}^1$ contains at least three elements.*

Proof. For any \mathcal{D} there is a natural homomorphism $G_{\mathcal{D}} \rightarrow \mathcal{S}_{|\mathcal{D}|}$, where $\mathcal{S}_{|\mathcal{D}|}$ is the symmetric group consisting of all permutations of the finite set $|\mathcal{D}|$.

If $\#|\mathcal{D}| \geq 3$, since any Möbius transformation which fixes three distinct points must be the identity, it follows that $G_{\mathcal{D}}$ maps isomorphically onto a subgroup of $\mathcal{S}_{|\mathcal{D}|}$.

Now suppose that $\#|\mathcal{D}| \leq 2$. After a Möbius transformation, we may assume that $|\mathcal{D}| \subset \{0, \infty\}$. (Here I am identifying the Riemann sphere with $\mathbb{C} \cup \{\infty\}$.) The group $G_{\mathcal{D}}$ then contains infinitely many transformations of the form

$$\mathbf{g}_{\kappa}(z) = \kappa z \quad \text{with} \quad \kappa \neq 0. \quad \square$$

The Moduli Space for Divisors.

6.

Let $\mathfrak{D}_n^{\text{fstab}}$ be the open subset of \mathfrak{D}_n consisting of all divisors with finite stabilizer (\iff all divisors with $\#\mathcal{D} \geq 3$).

Definition. The quotient $\mathfrak{M}_n = \mathfrak{D}_n^{\text{fstab}} / G$ will be called the **moduli space** for divisors, under the action of G .

Proposition 1. *This quotient space \mathfrak{M}_n is a T_1 -space, that is:*

*Every point of \mathfrak{M}_n is a closed subset,
 \iff Every G -orbit $((\mathcal{D})) = \{\mathbf{g}(\mathcal{D}) ; \mathbf{g} \in G\}$
in $\mathfrak{D}_n^{\text{fstab}}$ is closed as a subset of $\mathfrak{D}_n^{\text{fstab}}$.*

In other words, every $\mathcal{D}' \in \mathfrak{D}_n$ which belongs to the topological boundary $\overline{((\mathcal{D}))} \setminus ((\mathcal{D}))$ must have infinite stabilizer.

To prove Proposition 1, we must study elements of G which are “close to infinity” in G .

Distortion Lemma for Möbius Transformations.

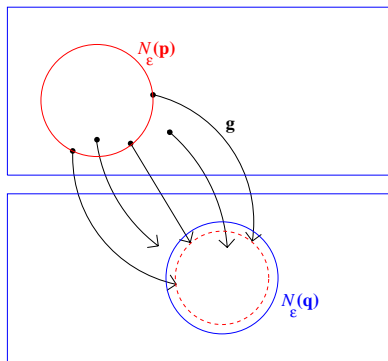
7.

Using the spherical metric on \mathbb{P}^1 , let $N_\varepsilon(\mathbf{p})$ be the open ε -neighborhood of \mathbf{p} .

Lemma. For any $\varepsilon > 0$
there is a large compact set
 $K = K_\varepsilon \subset G$

with the following property:
For any $\mathbf{g} \notin K$,
there are (not necessarily
distinct) points \mathbf{p} and \mathbf{q}
such that

$$\mathbf{g}(N_\varepsilon(\mathbf{p})) \cup N_\varepsilon(\mathbf{q}) = \mathbb{P}^1.$$



Thus points outside of $N_\varepsilon(\mathbf{p})$ map inside $N_\varepsilon(\mathbf{q})$.

(Proof Outline. The proof for the group of diagonal transformations $\mathbf{d}(x : y) = (\kappa x : y)$ is easy. But any $\mathbf{g} \in G$ can be written as a product $\mathbf{g} = \mathbf{r} \circ \mathbf{d} \circ \mathbf{r}'$ where \mathbf{r} and \mathbf{r}' are rotations of the Riemann sphere and \mathbf{d} is diagonal. ...)

Proof of Proposition 1: Points of \mathfrak{M}_n are closed.

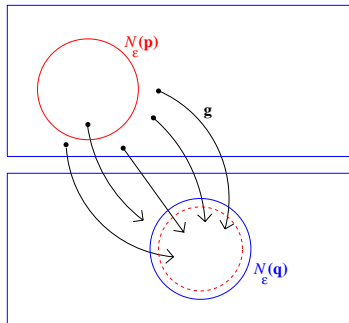
To prove: Every G -orbit $((\mathcal{D})) \subset \mathcal{D}_n^{\text{fstab}}$ is closed as a subset of $\mathcal{D}_n^{\text{fstab}}$.

Choose ε small enough so that any two points of $|\mathcal{D}|$ have distance $> 2\varepsilon$ from each other.

\implies No ε -ball contains more than one point of $|\mathcal{D}|$.

Given any $\mathbf{g} \notin K_\varepsilon$, choose \mathbf{p} and \mathbf{q} as in the Distortion Lemma. It follows that:

all but possibly one of the points of $\mathbf{g}(|\mathcal{D}|)$ lie in $N_\varepsilon(\mathbf{q})$.



Now suppose that we are given a sequence of points $\mathbf{g}_j(\mathcal{D}) \in ((\mathcal{D}))$ converging to $\mathcal{D}' \in \mathcal{D}_n$.

Case 1. If all $\mathbf{g}_j \in K \subset G$, then $\mathcal{D}' \in ((\mathcal{D}))$.

Case 2. If $\mathbf{g}_j \in K_{\varepsilon_j}$ with $\varepsilon_j \rightarrow 0$, then $|\mathcal{D}'|$ has at most two points, so $\mathcal{D}' \notin \mathcal{D}_n^{\text{fstab}}$. \square

The Cases $n \leq 4$ are very special.

9.

\mathfrak{M}_3 is a single point.

$\mathfrak{M}_4 \cong \mathbb{P}^1$ is a 2-sphere.

Proof Outline: Four distinct points in \mathbb{P}^1 determine a 2-fold branched covering which is an elliptic curve; characterized by the classical invariant $j(\mathcal{C}) \in \mathbb{C}$. Thus the open subset corresponding to divisors with four distinct points is canonically isomorphic to \mathbb{C} .

But there is one other G -orbit

$$((2\langle \mathbf{p} \rangle + \langle \mathbf{q} \rangle + \langle \mathbf{r} \rangle)) \subset \mathfrak{D}_4^{\text{fstab}}$$

consisting of divisors with only three distinct points.

It follows easily that \mathfrak{M}_4 is homeomorphic to the one point compactification $\mathbb{C} \cup \{\infty\} \cong \mathbb{P}^1$.

Theorem. For $n \geq 5$, \mathfrak{M}_n has a unique maximal open subset $\mathfrak{M}_n^{\text{Haus}}$ which is Hausdorff.

Here $\mathfrak{M}_n^{\text{Haus}}$ is the set of all images $\pi(\mathcal{D}) \in \mathfrak{M}_n$ where \mathcal{D} is a divisor with maximum multiplicity $m_{\max} < n/2$

(where $\pi : \mathfrak{D}_n^{\text{fstab}} \rightarrow \mathfrak{M}_n$ denotes the projection map).

$\mathfrak{M}_n^{\text{Haus}}$ is compact if n is odd;
but non-compact if n is even.

$\mathfrak{M}_n^{\text{Haus}}$ is an orbifold of complex dimension $n - 3$.

Points of \mathfrak{M}_n outside of $\mathfrak{M}_n^{\text{Haus}}$
are not even locally Hausdorff.

Definition. The action of a Lie group G on a space X is *proper* if, for every $x, y \in X$, there are neighborhoods U and V so that the set of group elements with $\mathbf{g}(U) \cap V \neq \emptyset$ has compact closure within G .

Standard Theorem. The quotient X/G of a Hausdorff space under a proper action is a Hausdorff space.

Using the Distortion Theorem, one can show that the action of $G(\mathbb{P}^1)$ on the space of divisors with $m_{\max} < n/2$ is proper.

To fix ideas, let $n = 5$. Consider two divisors of the form

$$\mathcal{D} = \mathcal{D}_2 + 3\langle \infty \rangle \quad \text{and} \quad \mathcal{D}' = \mathcal{D}_3 + 2\langle \infty \rangle$$

in \mathfrak{D}_5 , where

$$\mathcal{D}_2 = \langle \mathbf{p} \rangle + \langle \mathbf{q} \rangle \quad \text{and} \quad \mathcal{D}_3 = \langle \mathbf{p}' \rangle + \langle \mathbf{q}' \rangle + \langle \mathbf{r}' \rangle .$$

Let $\mathbf{g}_\kappa(z) = \kappa^2/z$, with $\kappa \gg 1$;

$$\text{so that } |z| < \kappa \iff |\mathbf{g}_\kappa(z)| > \kappa .$$

Then the two divisors $\mathcal{D}_2 + \mathbf{g}_\kappa(\mathcal{D}_3)$ and $\mathcal{D}_3 + \mathbf{g}_\kappa(\mathcal{D}_2)$ belong to the same G -orbit.

As $\kappa \rightarrow \infty$, the first converges to \mathcal{D}
and the second converges to \mathcal{D}' .

Thus every neighborhood of $\pi(\mathcal{D}) \in \mathfrak{M}_5$
intersects every neighborhood of $\pi(\mathcal{D}')$.

Since \mathcal{D}' can be arbitrarily close to \mathcal{D} , this proves that \mathfrak{M}_5 is not locally Hausdorff at the point $\pi(\mathcal{D})$.

Part 2. Curves in the Projective Plane.

13.

Definition. An **effective 1-cycle** of degree $n \geq 1$ on the complex projective plane \mathbb{P}^2 is a formal sum

$$\mathcal{C} = m_1 \cdot \mathcal{C}_1 + \cdots + m_k \cdot \mathcal{C}_k ,$$

where each \mathcal{C}_j is an irreducible complex curve, where the $m_j \geq 1$ are integers, and where $n = \sum_j m_j \deg(\mathcal{C}_j)$.

The space \mathfrak{C}_n of all effective 1-cycles can be given the structure of a complex projective space of dimension $n(n+3)/2$. (In fact each non-zero homogeneous polynomial $\Phi(x, y, z)$ of degree n has a zero locus consisting of irreducible curves \mathcal{C}_j , each counted with some multiplicity $m_j \geq 1$; yielding a 1-cycle.)

The group $G = G(\mathbb{P}^2) = \mathrm{PGL}_3(\mathbb{C})$ of all automorphisms of \mathbb{P}^2 acts on \mathbb{P}^2 and hence on the space \mathfrak{C}_n .

The stabilizer $G_{\mathcal{C}}$ of $\mathcal{C} \in \mathfrak{C}_n$ is just the group consisting of all projective automorphisms which map \mathcal{C} to itself.

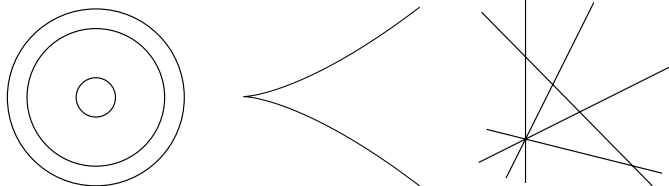
This stabilizer $G_{\mathcal{C}}$ may be either finite or infinite.

W-curves (and cycles).

14.

Curves with infinite stabilizer were first studied by Felix Klein and Sophus Lie, who called them **W-curves**.

Some examples:



Let $\mathfrak{W}_n \subset \mathfrak{C}_n$ be the algebraic set consisting of all cycles with infinite stabilizer. (\mathfrak{W}_n is a union of finitely many maximal irreducible subvarieties of \mathfrak{C}_n , of varying dimension.)

Note: \mathcal{C} has finite stabilizer if and only if the G -orbit $((\mathcal{C})) \subset \mathfrak{C}_n$ has dimension 8.

In fact $\dim((\mathcal{C})) + \dim(G_{\mathcal{C}}) = \dim(G) = 8$,
where $\dim(G_{\mathcal{C}}) = 0 \iff G_{\mathcal{C}}$ is finite.

The Moduli Space \mathbb{M}_n .

15.

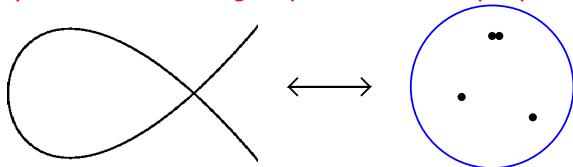
The complement $\mathfrak{C}_n^{\text{fstab}} = \mathfrak{C}_n \setminus \mathfrak{W}_n$ is the open set consisting of all cycles with **finite stabilizer**.

Definition. The quotient space $\mathbb{M}_n = \mathfrak{C}_n^{\text{fstab}}/G$, will be called the **moduli space** for plane cycles of degree n .

Examples. $\mathbb{M}_1 = \mathbb{M}_2 = \emptyset$.

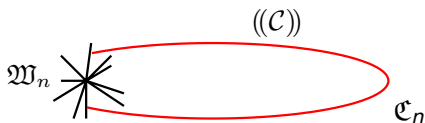
The moduli space \mathbb{M}_3 for cubic curves in \mathbb{P}^2 is canonically isomorphic to the moduli space \mathfrak{M}_4 for divisors in \mathbb{P}^1 .

Each has two “ramified points” corresponding to points with extra symmetry (= larger stabilizer). **Each also has one “improper point” where the group action is not proper.**



Thus $\mathbb{M}_3 \cong \mathbb{C} \cup \{\infty\} \cong \mathbb{P}^1$.

Cartoon of \mathfrak{C}_n , showing a typical G -orbit in red:



Theorem. The topological boundary of any G -orbit in \mathfrak{C}_n is contained in the closed subset \mathfrak{W}_n .

[Ghizzetti 1936; Aluffi and Faber 2010.]

\implies Every G -orbit of cycles with finite stabilizer is closed as a subset of $\mathfrak{C}_n^{\text{fstab}}$.

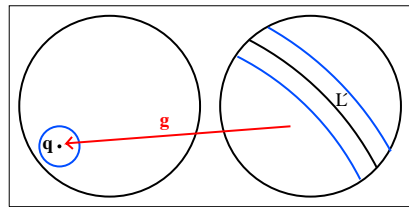
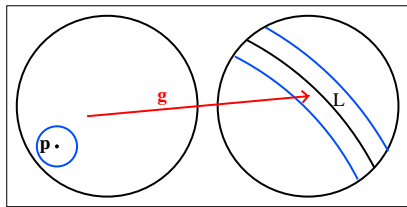
\implies Every point in \mathbb{M}_n is a closed set.

The Distortion Lemma for \mathbb{P}^2 .

17.

Lemma. Given $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset G(\mathbb{P}^2)$ with the following property.

For any $\mathbf{g} \notin K_\varepsilon$ there exists either:



(1) a point $\mathbf{p} \in \mathbb{P}^2$ and a line $L \subset \mathbb{P}^2$ such that $\mathbf{g}(N_\varepsilon(\mathbf{p})) \cup N_\varepsilon(L) = \mathbb{P}^2$

(so that \mathbf{g} maps every point outside of $N_\varepsilon(\mathbf{p})$ into $N_\varepsilon(L)$),

or (2) a line $L' \subset \mathbb{P}^2$ and a point $\mathbf{q} \in \mathbb{P}^2$ such that $\mathbf{g}(N_\varepsilon(L')) \cup N_\varepsilon(\mathbf{q}) = \mathbb{P}^2$

(so that \mathbf{g} maps every point outside of $N_\varepsilon(L')$ into $N_\varepsilon(\mathbf{q})$).

The Genus Invariant of a Singularity.

18.

Let \mathbf{p} be a singular point of a complex curve $\mathcal{C} \subset \mathbb{P}^2$. Let N_ϵ be a small round ball centered at \mathbf{p} .

If \mathcal{C}' is a smooth curve which closely approximates \mathcal{C} , then

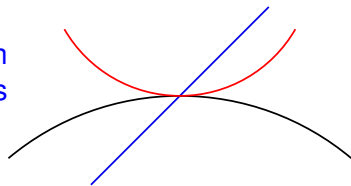
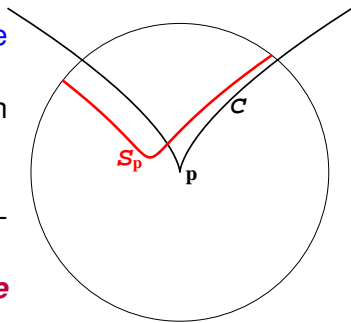
$$S_p = \mathcal{C}' \cap \overline{N_\epsilon}$$

is a compact connected Riemann-surface-with-boundary.

Its genus $g(S_p)$ will be called **the genus of the singularity $\mathbf{p} \in \mathcal{C}$** .

Examples: For a cusp singularity $x^p = y^q$ the genus is $(p-1)(q-1)/2$.

If \mathcal{C} is locally the union of k smooth branches \mathcal{B}_j , then the genus is $-1 + \sum_{i < j} \mathcal{B}_i \cdot \mathcal{B}_j$.



Two Properties of the Genus.

19.

Monotonicity. Suppose that

$$\mathcal{S} = \mathcal{S}_1 \cup \cdots \cup \mathcal{S}_k \subset \mathcal{S}' ,$$

where the \mathcal{S}_j are disjoint compact Riemann-surfaces-with-boundary, and \mathcal{S}' is another compact Riemann surface, possibly with boundary. Then

$$g(\mathcal{S}) := \sum g(\mathcal{S}_j) \leq g(\mathcal{S}) .$$

Scissors and Paste. Suppose that k disjoint embedded curves cut the closed Riemann surface \mathcal{S} into ℓ subspaces with boundary \mathcal{S}_j . Then

$$g(\mathcal{S}) = k + 1 - \ell + \sum g(\mathcal{S}_j) .$$

(This follows from the Euler characteristic identity

$$\chi(\mathcal{S}) = \sum \chi(\mathcal{S}_j) .)$$

A Hypothesis which implies Proper Action.

20.

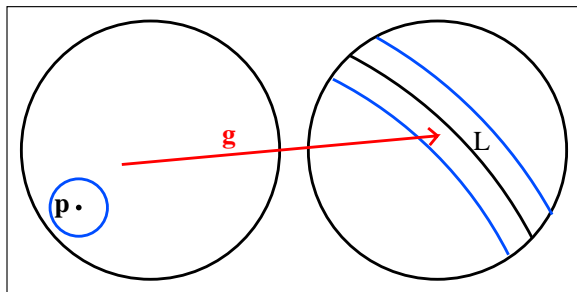
For any line $L \subset \mathbb{P}^2$ and any specified curve \mathcal{C} we can form the intersection $\mathcal{S}_L = \mathcal{C}' \cap \overline{N}_\varepsilon(L)$, where ε is small and \mathcal{C}' is a very close generic approximation to \mathcal{C} .

Lemma. *If*

$$g(\overline{\mathcal{C}' \setminus \mathcal{S}_p}) > g(\mathcal{S}_L)$$

for every $p \in |\mathcal{C}|$ and every $L \subset \mathbb{P}^2$,

then the action of G is locally proper at \mathcal{C} .



Let $\mathcal{S}_{\mathbf{p}}^* = \overline{\mathcal{C}' \setminus \mathcal{S}_{\mathbf{p}}}$. Then

$$\mathcal{S}_{\mathbf{p}} \cup \mathcal{S}_{\mathbf{p}}^* = \mathcal{C}' , \quad \mathcal{S}_{\mathbf{p}} \cap \mathcal{S}_{\mathbf{p}}^* = (\text{union of } k \text{ circles}) .$$

Therefore

$$\mathbf{g}(\mathcal{S}_{\mathbf{p}}) + \mathbf{g}(\mathcal{S}_{\mathbf{p}}^*) + k - \ell + 1 = \mathbf{g}(\mathcal{C}') = \binom{n-1}{2} .$$

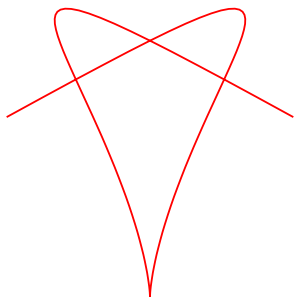
Here $\ell \geq 2$ is the number of components of $\mathcal{S}_{\mathbf{p}}$ plus the number of components of $\mathcal{S}_{\mathbf{p}}^*$.

Define the **augmented genus** of $\mathcal{S}_{\mathbf{p}}$ to be

$$\mathbf{g}^+(\mathcal{S}_{\mathbf{p}}) = \mathbf{g}(\mathcal{S}_{\mathbf{p}}) + k - 1 .$$

Together with the Lemma, this formula yields:








Theorem. *If $\mathbf{g}^+(\mathcal{S}_{\mathbf{p}}) + \mathbf{g}(\mathcal{S}_L) < \mathbf{g}(\mathcal{C}')$ for every $\mathbf{p} \in \mathcal{C}$ and every $L \subset \mathbb{P}^2$, then the action of G is locally proper at \mathcal{C} .*



Let $\mathcal{U}_n \subset \mathcal{C}_n$ be the open set consisting of curves with no singularities other than simple double points and cubic cusps.

Corollary. *If $n \geq 4$ then the action of $G(\mathbb{P}^2)$ is locally proper throughout \mathcal{U}_n .*

In fact the action is proper throughout \mathcal{U}_N , so the quotient space $\mathcal{U}_n/G(\mathbb{P}^2) \subset \mathbb{M}_n$ is a Hausdorff space.

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