

Understanding Cubic Maps

John Milnor

Stony Brook University (www.math.sunysb.edu)

TOULOUSE, DECEMBER 2009

PROBLEM: To study cubic polynomial maps F with a critical point which is periodic under F .

—work with Araceli Bonifant and Jan Kiwi—

Normal form:

Any cubic polynomial map is affinely conjugate to a monic centered map

$$F(z) = F_{a,v}(z) = (z - a)^2(z + 2a) + v.$$

Here a is the **marked critical point**,
and $F(a) = v$ is the **marked critical value**.

The **parameter space** for the family of all such maps is the set of all pairs $(a, v) \in \mathbb{C}^2$.

The Period p Curve

2.

Definition. The **period p curve** \mathcal{S}_p consists of those parameter pairs $(a, v) \in \mathbb{C}^2$ such that that marked critical point a for $F = F_{a,v}$ has period *exactly* p .

(Conjecture: \mathcal{S}_p is irreducible for all $p \geq 1$.)

Degree computation: The set of parameter pairs (a, v) which satisfy the polynomial equation

$$F^{\circ p}(a) = a \tag{1}$$

forms a smooth affine variety

$$\mathcal{S}_p^{\oplus} = \bigsqcup_{n|p} \mathcal{S}_n \subset \mathbb{C}^2.$$

Equation (1) has degree 3^{p-1} . Hence the degree d_p of \mathcal{S}_p can be computed inductively from the equation

$$\sum_{n|p} d_n = 3^{p-1}.$$

$$d_1 = 1, \quad d_2 = 2, \quad d_3 = 8, \quad d_4 = 24, \quad d_5 = 80, \quad \dots$$

Canonical Coordinates for \mathcal{S}_p

3.

Define $H_p : \mathbb{C}^2 \rightarrow \mathbb{C}$ by

$$H_p(a, v) = F^{\circ p}(a) - a, \quad \text{with } F = F_{a,v}.$$

This vanishes everywhere on \mathcal{S}_p , with $dH_p \neq 0$ on \mathcal{S}_p .

Think of H_p as a complex Hamiltonian function, and consider the Hamiltonian differential equation

$$\frac{da}{dt} = \frac{\partial H_p}{\partial v}, \quad \frac{dv}{dt} = -\frac{\partial H_p}{\partial a}.$$

The local solutions $t \mapsto (a, v) = (a(t), v(t))$ are holomorphic, and lie in curves $H_p = \text{constant}$.

Those solutions which lie in \mathcal{S}_p provide a local holomorphic parametrization, unique up to a translation, $t \mapsto t + \text{constant}$.

Equivalently, the holomorphic 1-form

$$dt = \frac{da}{\partial H_p / \partial v} \quad \text{and/or} \quad \frac{-dv}{\partial H_p / \partial a}$$

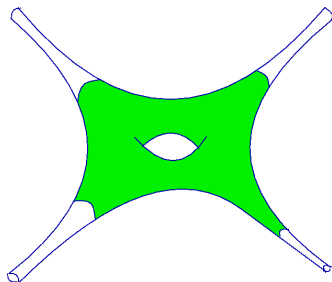
is well defined and non-zero everywhere on \mathcal{S}_p .

Smooth compactification

4.

More generally, any smooth affine curve $\mathcal{S} \subset \mathbb{C}^2$ has such a canonical 1-form dt .

Such a curve can be decomposed (non-uniquely) into a compact subset, together with finitely many end regions \mathcal{E}_h , each conformally isomorphic to $\mathbb{C} \setminus \overline{\mathbb{D}}$.



We can compactify, to obtain a smooth compact complex 1-manifold $\overline{\mathcal{S}}$, by adding a single ideal point ∞_h to each end region \mathcal{E}_h .

The holomorphic 1-form dt on \mathcal{S} becomes a meromorphic 1-form on $\overline{\mathcal{S}}$, with zeros or poles only at the ideal points.

The **Euler characteristic** of $\overline{\mathcal{S}}$ can be computed as follows:

$$\chi(\overline{\mathcal{S}}) = \#(\text{poles}) - \#(\text{zeros}),$$

counted with multiplicity.

If \mathcal{S} is connected, then

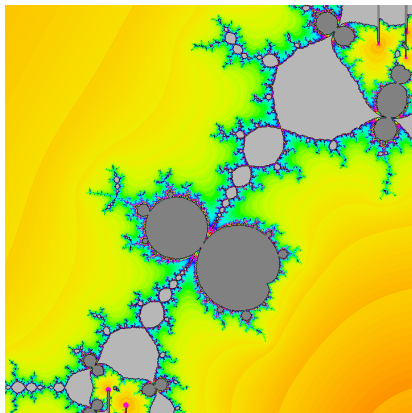
$$\text{genus}(\mathcal{S}) = \text{genus}(\overline{\mathcal{S}}) = 1 - \chi(\overline{\mathcal{S}})/2.$$

Special properties of the period p curve

6.

There is a dynamically defined compact subset of \mathcal{S}_p , namely the **connectedness locus** $\mathcal{C}(\mathcal{S}_p)$ consisting of all maps $F \in \mathcal{S}_p$ such that the Julia set $J(F)$ is connected.

Each connected component $\mathcal{E}_h \subset \mathcal{S}_p \setminus \mathcal{C}(\mathcal{S}_p)$, called an **escape region** in \mathcal{S}_p , is conformally isomorphic to $\mathbb{C} \setminus \overline{\mathbb{D}}$.



The winding number

7.

Theorem. The residue of dt at each ideal point $\infty_h \in \overline{S}_p$ is zero:

$$\frac{1}{2\pi i} \oint_{\infty_h} dt = 0.$$

Thus t can be defined as a meromorphic function throughout any simply connected subset of \overline{S}_p .

Normal form near an ideal point ∞_h : We can choose a local parameter ζ for \overline{S}_p , and a canonical parameter t , so that

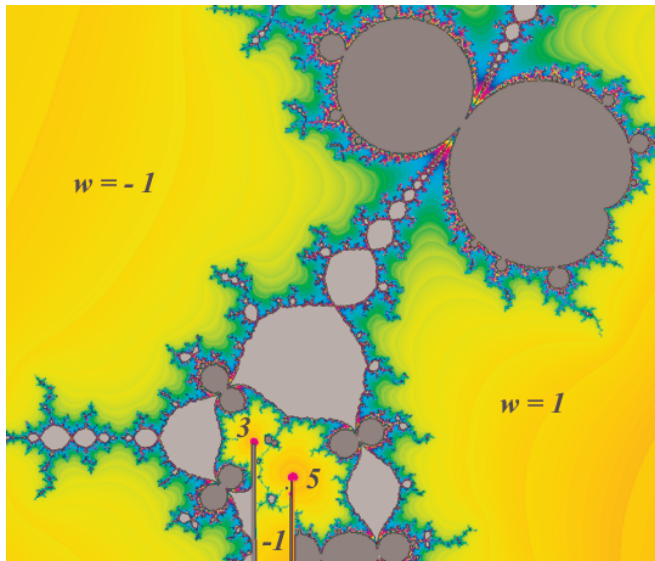
$$t = \zeta^{w_h}, \quad \text{with} \quad w_h \in \mathbb{Z}, \quad w_h \neq 0.$$

Here w_h is the **winding number** of the t -plane around ∞_h .

$$\text{As } \zeta \rightarrow 0, \quad \text{note that } t \rightarrow \begin{cases} 0 & \text{if } w > 0, \\ \infty & \text{if } w < 0. \end{cases}$$

Winding number: examples in \mathcal{S}_4

8.



Euler characteristic formulas

9.

Since $t = \zeta^{w_h}$,

$$dt = d(\zeta^{w_h}) = w_h \zeta^{w_h-1} d\zeta,$$

with a zero of order $w_h - 1$ at the ideal point.

Thus the formula $\chi = \#(\text{poles}) - \#(\text{zeros})$ takes the form

$$\chi(\bar{\mathcal{S}}_p) = \sum_h (1 - w_h),$$

summed over all ideal points. With a lot of work, this yields

$$\chi(\bar{\mathcal{S}}_p) = (2 - p)d_p + (\text{number of ideal points}).$$

(Key tool for the proof:

Kiwi's theory of dynamics, including Branner-Hubbard puzzles, over the completion of the field of formal Puiseux series.)

Examples:

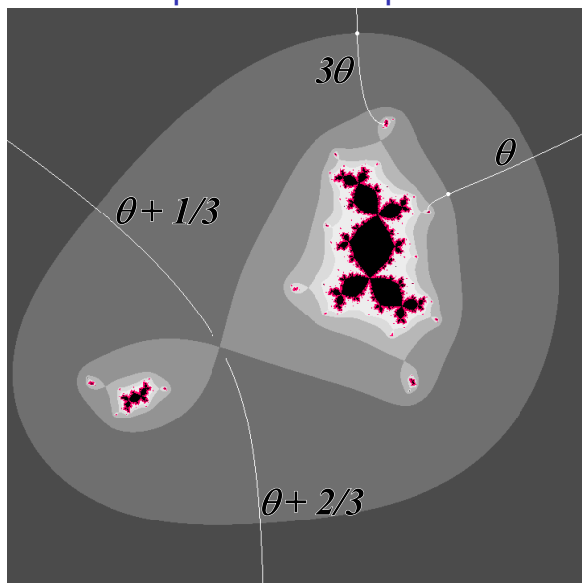
$$\chi(\mathcal{S}_1) = \chi(\mathcal{S}_2) = 2, \quad \chi(\mathcal{S}_3) = 0, \quad \chi(\mathcal{S}_4) = -28.$$

Further computations of $\chi(\overline{S}_\rho)$ by Laura DeMarco 10.

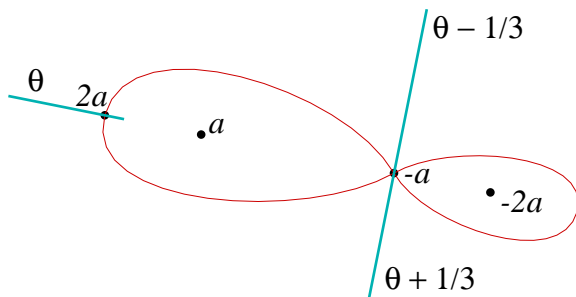
- ▶ Period 5: -184
- ▶ Period 6: -784
- ▶ Period 7: -3236
- ▶ Period 8: -11848
- ▶ Period 9: -42744
- ▶ Period 10: -147948
- ▶ Period 11: -505876
- ▶ Period 12: -1694848
- ▶ Period 13: -5630092
- ▶ Period 14: -18491088
- ▶ Period 15: -60318292
- ▶ Period 16: -195372312
- ▶ Period 17: -629500300
- ▶ Period 18: -2018178780
- ▶ Period 19: -6443997852
- ▶ Period 20: -20498523320

Dynamics: A sample Julia set picture

11.



Filled Julia set for a map in the “rabbit” escape region of \mathcal{S}_3 .



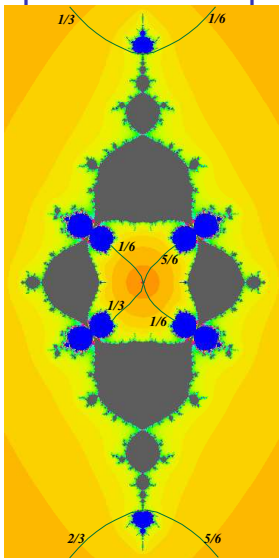
Sketch of the dynamic plane for a map belonging to any escape region $\mathcal{E}_h \subset \mathcal{S}_p$.

Critical points: $a, -a$.

Cocritical points: $2a, -2a$, with $F(\pm 2a) = F(\mp a)$.

Definition: $\theta = \theta(F) \in \mathbb{R}/\mathbb{Z}$ is the **cocritical angle**.

Rays in parameter space: Examples in \mathcal{S}_2 13.



The indicated rays all land at parabolic maps,
and have angles of the form $m/3n$.

Each external ray in an escape region $\mathcal{E}_h \subset \mathcal{S}_p$ is labeled by its cocritical angle $\theta(F) \in \mathbb{R}/\mathbb{Z}$.

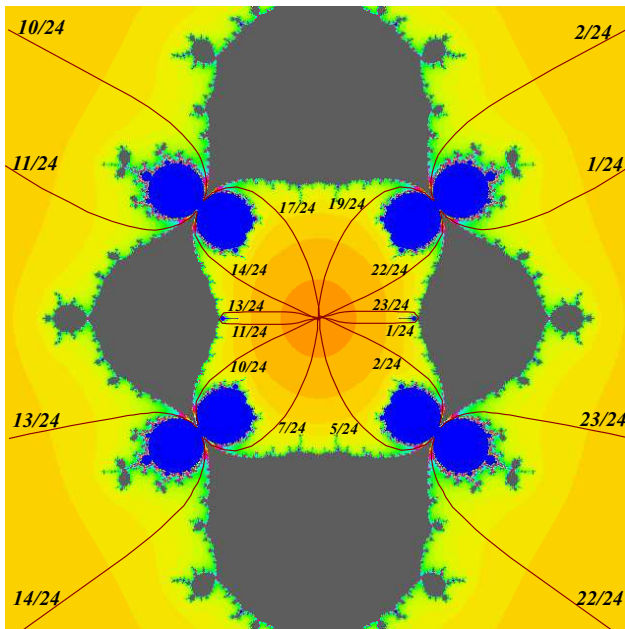
Theorem. *Every parameter ray with rational cocritical angle θ lands at a well defined map F_0 in the topological boundary $\partial\mathcal{E}_h \subset \mathcal{S}_p$.*

This landing map F_0 has a parabolic orbit

\iff one of the two angles $\theta \pm 1/3$ is periodic.

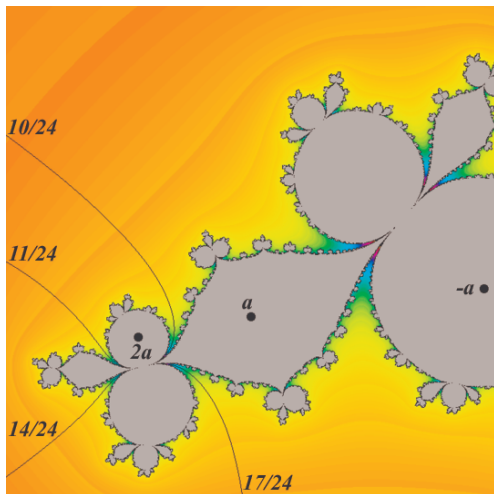
\iff θ has the form $\frac{m}{3n}$ with $3 \nmid m$ and $3 \nmid n$.

Complication: For each θ , there are μ_h distinct parameter rays in \mathcal{E}_h with label θ , where $\mu_h \geq 1$ is an invariant called the **multiplicity** of \mathcal{E}_h .



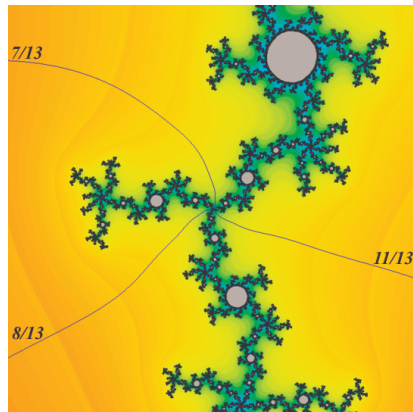
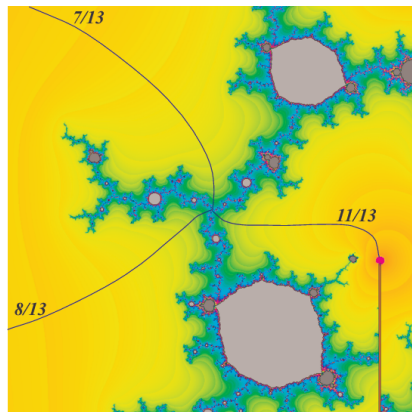
Picture of a corresponding of a Julia set $J(F_0)$

16.



Here F_0 is the landing map for the $10/24$, $11/24$, $14/24$, and $17/24$ rays at the upper left of the previous figure.

Theorem: If the landing map for a rational parameter ray is not parabolic, then it is critically finite. An example in \mathcal{S}_4 :



The same rays land at $F \in \mathcal{S}_p$ as at $2a_F \in J(F)$.

Asymptotic similarity (as in Tan Lei)

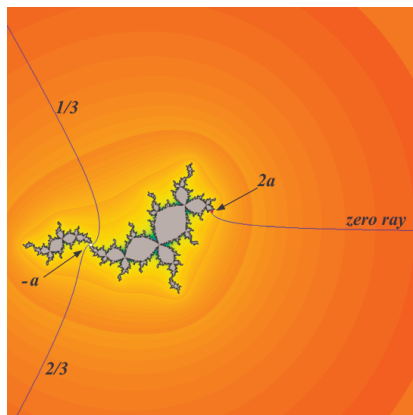
18.

$F \in \mathcal{S}_\rho$ critically finite map, $\eta =$ multiplier of postcritical cycle.

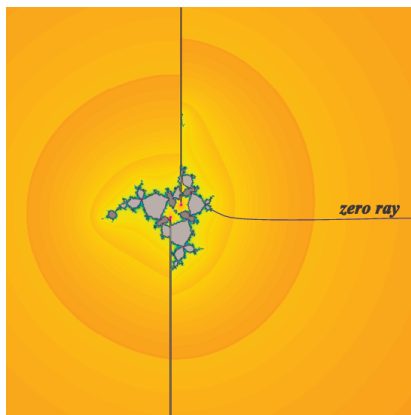
Kœnigs: There is a Hausdoff limit $\lim_{n \rightarrow \infty} \eta^n (K(F) - 2a)$.

Linear equivalence: $\cong \lim_{n \rightarrow \infty} \eta^n (\mathcal{C}(\mathcal{S}_\rho) - F)$



(interpreting last expression using a local parameter).



Julia set



parameter space (in \mathcal{S}_3)

-  J. Kiwi, *Puiseux series polynomial dynamics and iteration of complex cubic polynomials*, *Ann. Inst. Fourier (Grenoble)* **56** (2006) 1337–1404.
-  *Cubic Polynomial Maps with Periodic Critical Orbit:*
- Part I*, in “Complex Dynamics Families and Friends”, ed. D. Schleicher, A. K. Peters 2009, pp. 333-411.
- Part II: Escape Regions* (with Bonifant and Kiwi), arXiv:0910.1866. To appear in *Journal of Conformal Geometry and Dynamics*.
- Part III: External rays* (with Bonifant), in preparation.