

Critically Periodic Cubic Polynomials

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IN MEMORY OF ADRIEN DOUADY

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Parameter Space

1.

THE PROBLEM: To study cubic polynomial maps F with a marked critical point which is periodic under F .

—work in progress with Araceli Bonifant—

Any cubic polynomial map with marked critical point is affinely conjugate to one of the form

$$F(z) = F_{a,v}(z) = z^3 - 3a^2z + (2a^3 + v).$$

Here a is the **marked critical point**,
 $F(a) = v$ is the **marked critical value**,
 $-a$ is the **free critical point**.

*The set of all such maps $F = F_{a,v}$ will be identified with the **parameter space**, consisting of all pairs $(a, v) \in \mathbb{C}^2$.*

The Period p Curve

2.

Definition: the **period p curve** $\mathcal{S}_p \subset \mathbb{C}^2$, consists of all maps $F = F_{a,v}$ such that the marked critical point a has period exactly p .

Assertion. \mathcal{S}_p is a smooth affine curve in \mathbb{C}^2 .

Complication: The genus of \mathcal{S}_p increases rapidly with p .

\mathcal{S}_1 has genus zero with one puncture ($\cong \mathbb{C}$),

\mathcal{S}_2 has genus zero with two punctures,

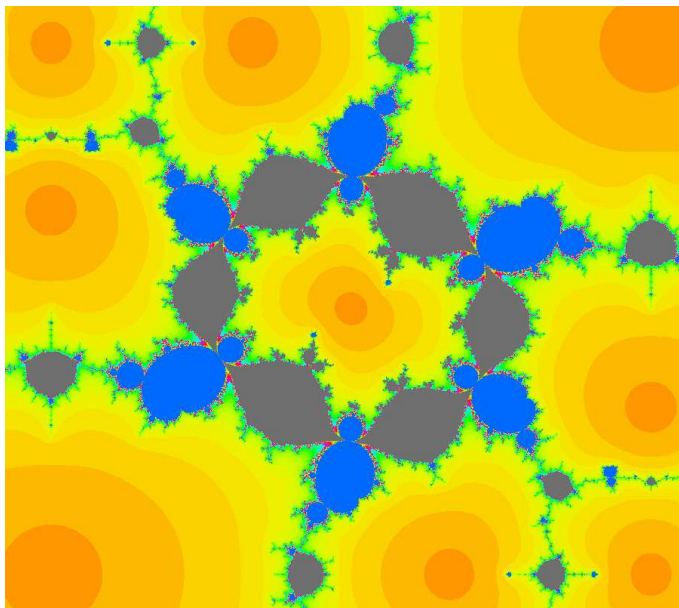
\mathcal{S}_3 has genus one with 8 punctures,

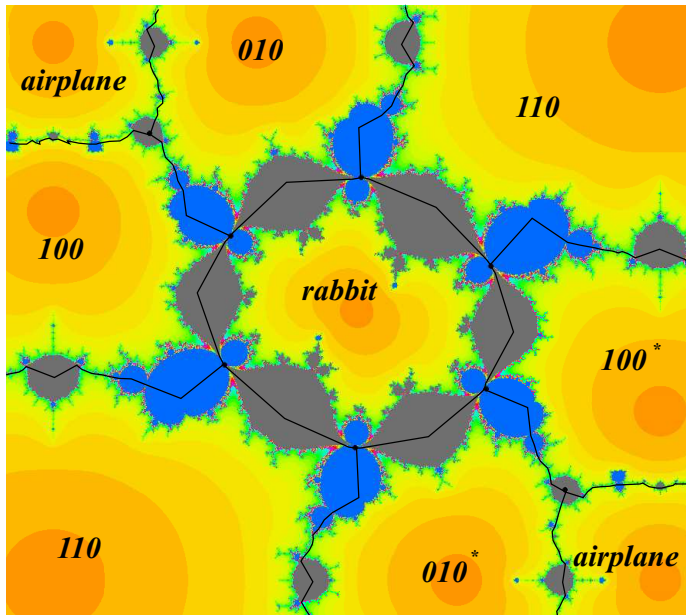
\mathcal{S}_4 has genus 15 with 20 punctures, \dots

We can simplify a little by passing to to the **moduli space** $\mathcal{S}_p/\mathcal{I}$ of holomorphic conjugacy classes. Here \mathcal{I} is the involution

$$F(z) \leftrightarrow -F(-z), \quad \text{so that} \quad F_{a,v} \leftrightarrow F_{-a,-v}.$$

The genus of $\mathcal{S}_p/\mathcal{I}$ is smaller, but still increases with p .





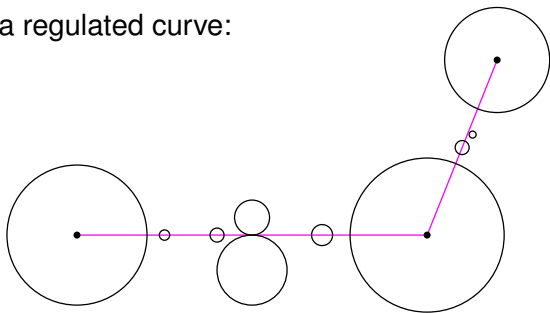
A Cell Structure in $\overline{\mathcal{S}}_p$.

5.

Let $\overline{\mathcal{S}}_p$ be the smooth compact surface obtained from \mathcal{S}_p by filling in each puncture point.

Conjecture. *There is a canonical cell subdivision of each $\overline{\mathcal{S}}_p$. For $p \geq 2$, the 1-skeleton can be identified with the union of all simple closed regulated curves.*

Sketch of a regulated curve:



Escape Regions

6.

Let $\mathcal{C}(\mathcal{S}_p)$ be the **connectedness locus** in \mathcal{S}_p .

Each connected component \mathcal{E} of the complement $\mathcal{S}_p \setminus \mathcal{C}(\mathcal{S}_p)$ will be called an **escape region** in \mathcal{S}_p .

Theorem. *For each \mathcal{E} , there is a canonical covering map*

$$\mathcal{E} \rightarrow \mathbb{C} \setminus \overline{\mathbb{D}}.$$

The degree of this covering map will be called the **multiplicity** $\mu \geq 1$ of the escape region.

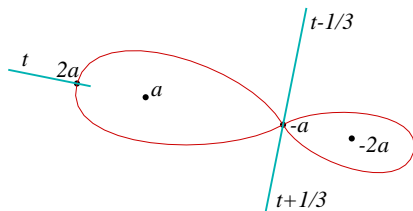
We can talk about **equipotentials** and **parameter rays** in each escape region.

Notation: *A parameter ray in the escape region \mathcal{E} will be denoted by $\mathcal{R}_{\mathcal{E}}(t)$. Here $t \in \mathbb{R}/\mu\mathbb{Z}$.*

If $\mu > 1$, then t will be called a **generalized angle**.

The Dynamic Plane for a map $F \in \mathcal{E}$.

7.



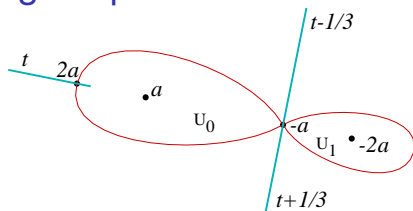
For F in the escape region \mathcal{E} , the equipotential through $2a$ and $-a$ is a figure eight curve. Here $2a$ is the free **cocritical point**, with $F(2a) = F(-a)$.

The Böttcher coordinate $\beta(2a) \in \mathbb{C} \setminus \overline{\mathbb{D}}$ of the escaping cocritical point is well defined, and the correspondence $F \mapsto \beta(2a)$ is the required covering map

$$\mathcal{E} \rightarrow \mathbb{C} \setminus \overline{\mathbb{D}}.$$

The Kneading Sequence.

8.



Let U_0 and U_1 be the two bounded regions cut out by the figure eight curve, with $a \in U_0$. Any bounded orbit $z_1 \mapsto z_2 \mapsto \dots$ determines a sequence $\sigma_1, \sigma_2, \dots$ of zeros and ones with

$$z_j \in U_{\sigma_j}.$$

Now take z_1 equal to the marked critical value $v = F(a)$.

The associated sequence $\{\sigma_j\}$ will be called the **kneading sequence** of the escape region \mathcal{E} . Thus

$$F^{\circ j}(a) \in U_{\sigma_j} \quad \text{for } j \geq 1.$$

The Associated Quadratic Map.

9.

The kneading sequence of any escape region $\mathcal{E} \subset \mathcal{S}_p$ is clearly periodic: its period p_1 divides p .

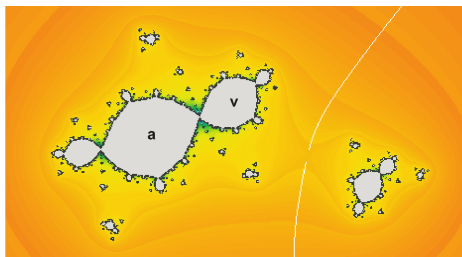
Theorem (Branner and Hubbard). *Suppose that F belongs to the escape region $\mathcal{E} \subset \mathcal{S}_p$. Then the Julia set $J(F)$ consists of countably many copies of a quadratic Julia set $J(Q)$, together with uncountably many single point components. Here the quadratic polynomial $Q = Q_{\mathcal{E}}$ is critically periodic of period p_2 where*

$$p = p_1 p_2 .$$

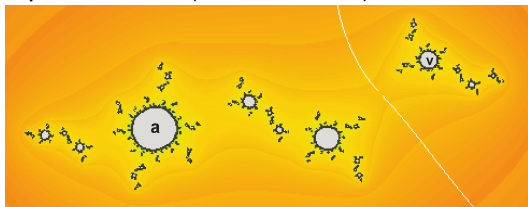
In other words:

Period of marked critical point

= (kneading period) \times (associated quadratic period).



Here the kneading sequence is $\overline{00}$, and the associated quadratic map is $z^2 - 1$ (the “basilica”).



Kneading sequence $\overline{10}$, with associated quadratic z^2 .

Canonical Coordinates for \mathcal{S}_p .

11.

Consider the function

$$H_p : \mathbb{C}^2 \rightarrow \mathbb{C}, \quad H_p(a, v) = F_{a,v}^{\circ p}(a) - a$$

which vanishes everywhere on \mathcal{S}_p . Think of H_p as a “complex Hamiltonian function”, and consider the Hamiltonian differential equation

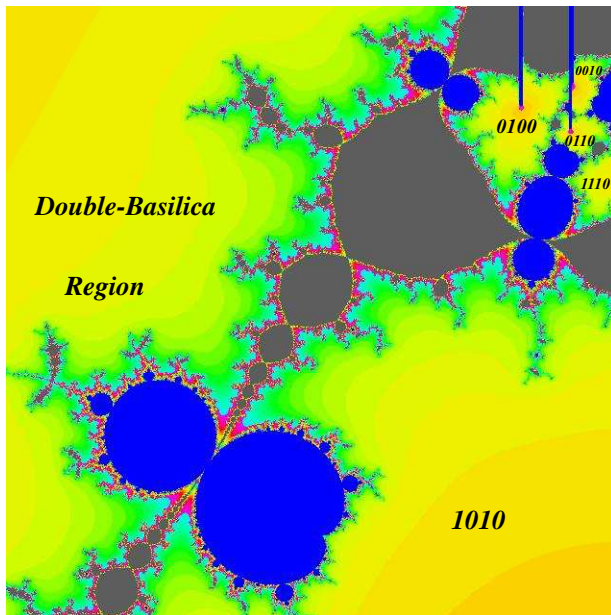
$$\frac{da}{dt} = \frac{\partial H_p}{\partial v}, \quad \frac{dv}{dt} = -\frac{\partial H_p}{\partial a}.$$

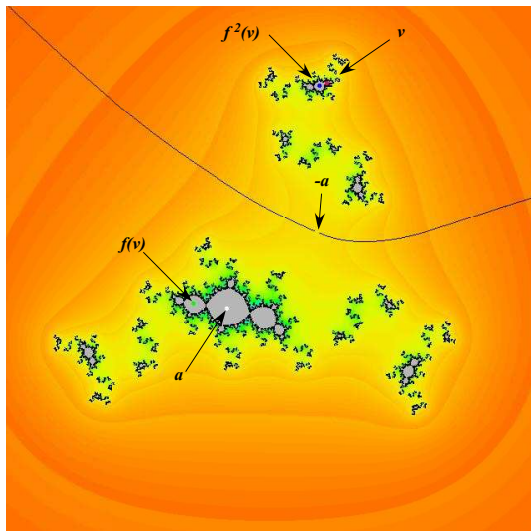
There are holomorphic local solutions

$$t \mapsto (a, v) = \Phi(t).$$

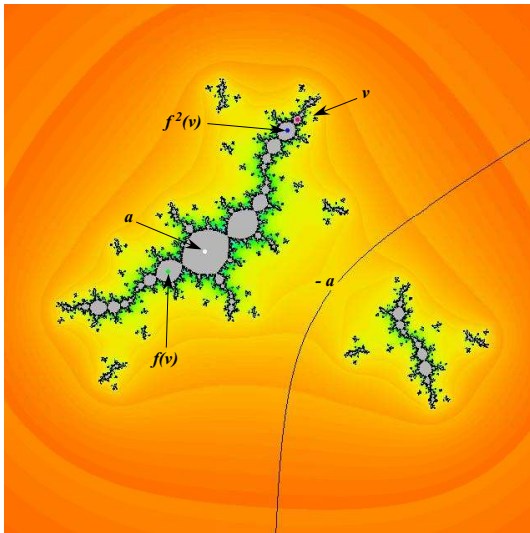
These lie in curves $H_p = \text{constant}$, parallel to \mathcal{S}_p . Those solutions which lie in \mathcal{S}_p provide a local holomorphic parametrization, unique up to translation of the t -coordinate.

Equivalent description: *There is a canonical 1-form dt which is well defined and non-zero throughout \mathcal{S}_p .*

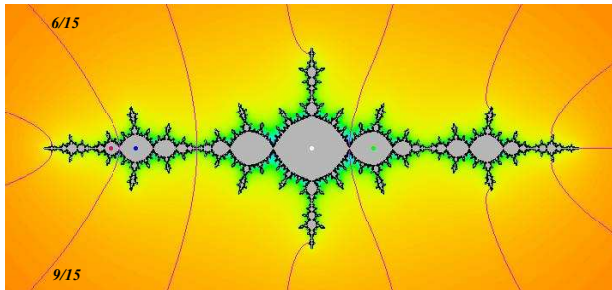




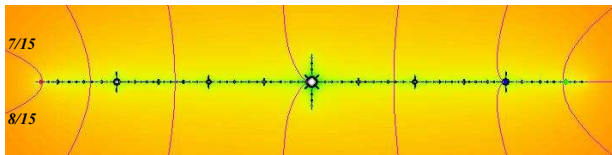
Kneading sequence $1010\dots$, with period $p_1 = 2$.
 $Q(z) = z^2 - 1$ with critical period $p_2 = 2$.



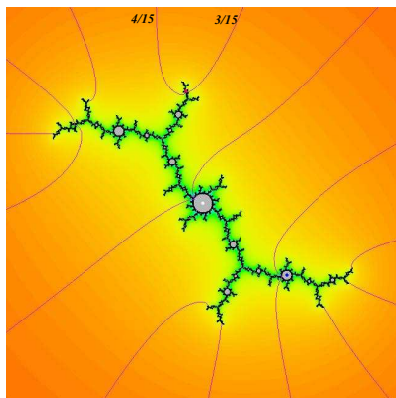
Kneading sequence $0000\dots$, with period $p_1 = 1$.
 $Q(z) = z^2 - 1.3107\dots$ with critical period $p_2 = 4$.



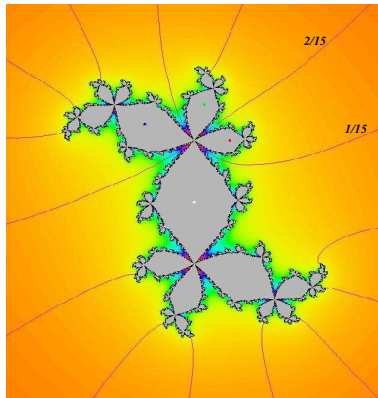
Double-Basilica



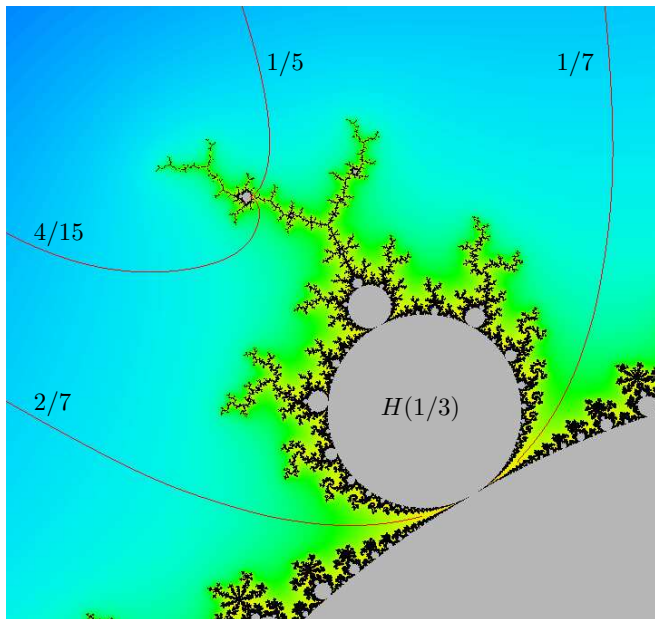
Worm



Kokopelli



$(1/4)$ -Rabbit



Let $\mathcal{E} \subset \mathcal{S}_p$ be any escape region.

Theorem. *If the generalized angle t_0 is rational, then the ray $\mathcal{R}_{\mathcal{E}}(t_0)$ lands at a well defined point F_0 in the boundary $\partial\mathcal{E}$. Furthermore, F_0 is either critically finite, or parabolic.*

Define $t \in \mathbb{Q}/\mathbb{Z}$ to be **co-periodic** if:

$t \pm 1/3$ is periodic under angle tripling,

$\Leftrightarrow 3t$ is periodic but t is not periodic,

$\Leftrightarrow t$ has the form $\frac{m}{3n}$ where m and n are not divisible by 3.

Theorem. *If $t_0 \pmod{\mathbb{Z}}$ is co-periodic, then the landing point of $\mathcal{R}_{\mathcal{E}}(t_0)$ is parabolic.*

We believe that this should be an if and only if statement:

t_0 co-periodic \Leftrightarrow the landing point is parabolic.

If $t \pm 1/3$ has period q , we say that t has **co-period** q .

Note that any angle of co-period q can be written as a fraction

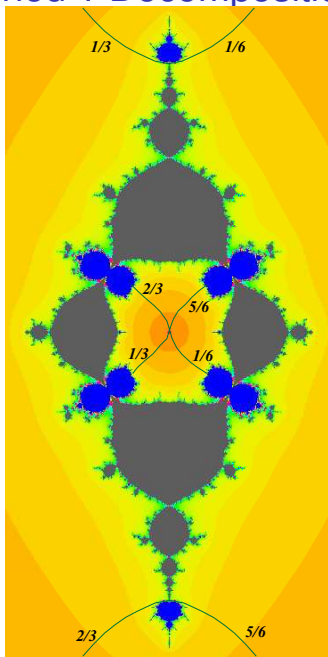
$$t = \frac{m}{3(3^q - 1)}.$$

For example, $q = 1 \Rightarrow t = m/6,$
 $q = 2 \Rightarrow t = m/24.$

Period q decomposition: *The collection of all rays of co-period q , together with their landing points, decomposes the parameter curve S_ρ into a finite number of connected open sets U_j .*

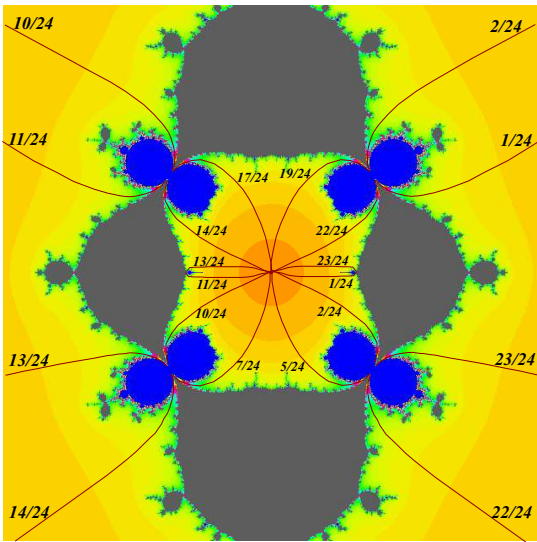
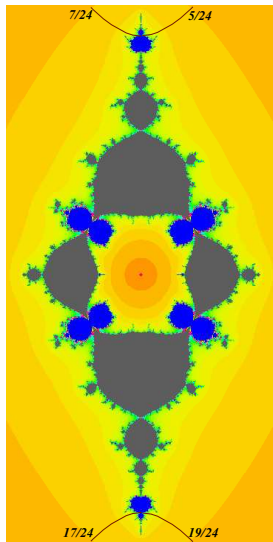
Example: The Period 1 Decomposition of \mathcal{S}_2 .

20.



Period 2 Decomposition of S_2 .

21.



Stability of Periodic Orbits.

22.

Let U_j be any connected component of

$$\mathcal{S}_p \setminus \overline{\bigcup \text{rays of coperiod } q},$$

and let $t_0 \in \mathbb{Q}/\mathbb{Z}$ have period q .

As F varies over U_j , the dynamic ray $\mathcal{R}_F(t_0)$ varies smoothly:

Theorem. *For each $F \in U_j$, and each angle $t_0 \in \mathbb{Q}/\mathbb{Z}$ of period q , the ray $\mathcal{R}_F(t_0)$ lands at a repelling periodic point $z_F \in J(F) \subset \mathbb{C}$.*

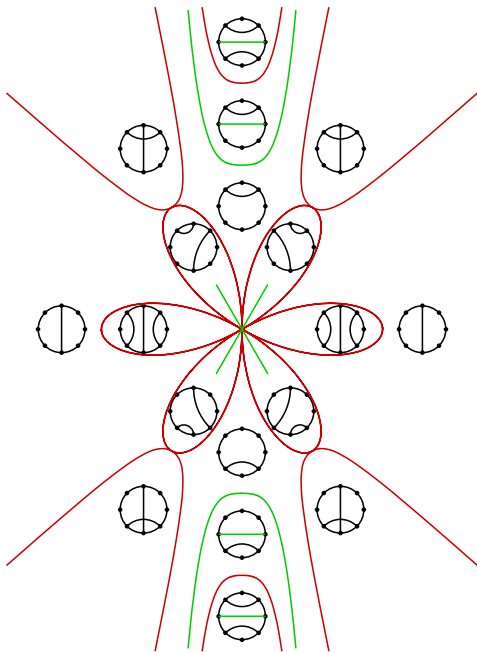
Furthermore, the correspondence $F \mapsto z_F$ defines a holomorphic function $U_j \rightarrow \mathbb{C}$.

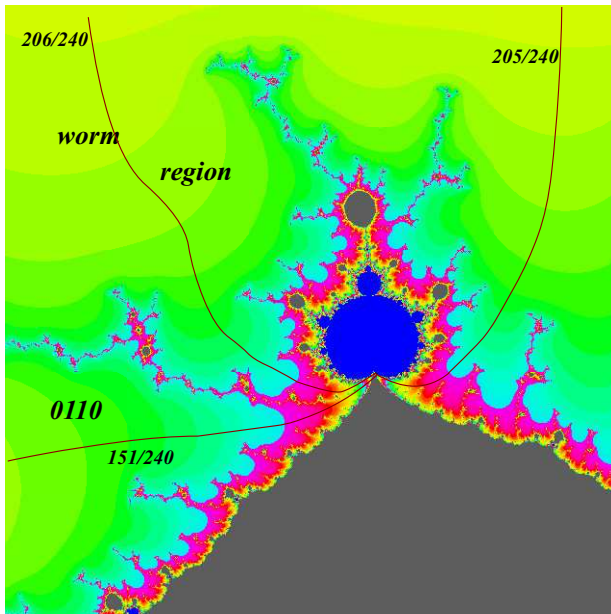
The pattern of which dynamic rays of period q have a common landing point is the same for all $F \in U_j$.

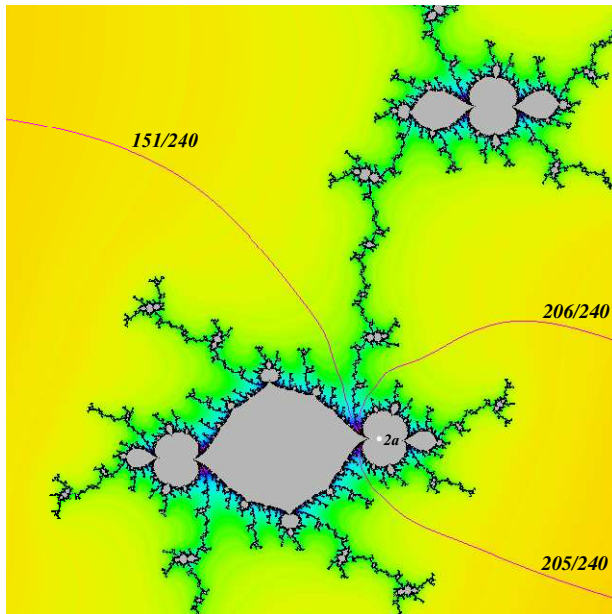
Corollary. Every parabolic map $F_0 \in \mathcal{S}_p$ is the landing point of at least one co-periodic ray.

Orbit Portraits for $F \in \mathcal{S}_2$ (Periods 1 and 2).

23.





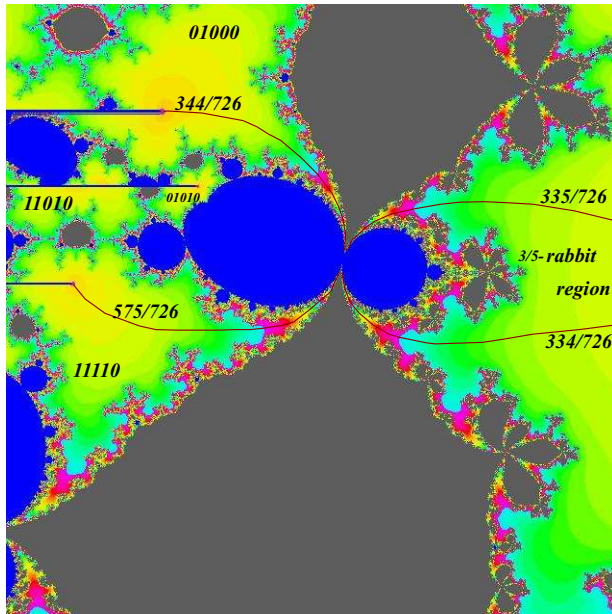


(Empirical Claims)

Every Mandelbrot component $\mathcal{M} \subset \mathcal{S}_p$ has a well defined root point F_0 , and every parabolic point $F_0 \in \mathcal{S}_p$ is the root point of a unique Mandelbrot component $\mathcal{M} \subset \mathcal{S}_p$.

For $F \in \mathcal{M}$, let r_0 be the root point of the Fatou component $U(2a)$ containing the cocritical point $2a$. Then a neighborhood of F_0 in \mathcal{S}_p is closely related to a neighborhood of r_0 in the dynamic plane for F . More precisely:

- The two closest parameter rays at F_0 which enclose \mathcal{M} have the same angles (modulo \mathbb{Z}) as the two closest dynamic rays at r_0 which enclose $U(2a)$.
- Furthermore, any parameter ray landing at F_0 has the same angle (modulo \mathbb{Z}) as some dynamic ray landing at r_0 .



Detail of corresponding Julia Set

28.

