

Cubic Polynomial Maps

with periodic critical orbit

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Parameter Space

THE PROBLEM: To study cubic polynomial maps f with a marked critical point which is periodic under f .

Any cubic polynomial map with marked critical point is affinely conjugate to one of the form

$$f(z) = f_{a,v}(z) = z^3 - 3a^2z + (2a^3 + v),$$

with critical points $\pm a$.

Here a is the **marked critical point**, and $f(a) = v$ is the **marked critical value**.

*The **parameter space** for this family consists of all pairs $(a, v) \in \mathbb{C}^2$.*

Alternative expression: $f(z) = (z - a)^2(z + 2a) + v$.

Moduli Space

This normal form

$f_{a,v}(z) = z^3 - 3a^2z + (2a^3 + v)$ is *almost* unique.

However, $f_{a,v}$ is affinely conjugate to the map

$$f_{-a,-v}(z) = -f_{a,v}(-z),$$

with Julia set (in the z -plane) rotated by 180° .

Form the quotient of the parameter plane \mathbb{C}^2 by the involution

$$\mathcal{I} : (a, v) \mapsto (-a, -v).$$

Definition. This quotient \mathbb{C}^2/\mathcal{I} will be identified with the **moduli space**, consisting of all affine conjugacy classes of marked cubic maps.

The Period p Curve

Definition: the **period p curve** $\mathcal{S}_p \subset \mathbb{C}^2$, consists of all pairs (a, v) such that the marked critical point of $f_{a,v}$ has period exactly p . **FOUR BASIC FACTS:**

1. This period p curve \mathcal{S}_p is a smooth affine curve in the (a, v) -coordinate space \mathbb{C}^2 . Its quotient $\mathcal{S}_p/\mathcal{I}$ is a smooth curve in the moduli space \mathbb{C}^2/\mathcal{I} .

2. \mathcal{S}_p can be compactified by adding finitely many **ideal points**, thus yielding a compact complex 1-manifold $\overline{\mathcal{S}}_p$. Similarly $\overline{\mathcal{S}}_p/\mathcal{I}$ is a compact complex 1-manifold with finitely many ideal points.

(CAUTION: $\overline{\mathcal{S}}_p$ is NOT the closure of \mathcal{S}_p in projective space.)

Definition. *The **connectedness locus** $\mathcal{C}(\mathcal{S}_p)$ consists of all maps in \mathcal{S}_p with connected Julia set.*

3. This connectedness locus is a compact subset of \mathcal{S}_p .

Escape Regions

4. Each connected component of the complement $\overline{\mathcal{S}_p} \setminus \mathcal{C}(\mathcal{S}_p)$ is conformally isomorphic to the open unit disk, **with an ideal point at its center.**

*Such components will be called **escape regions.***

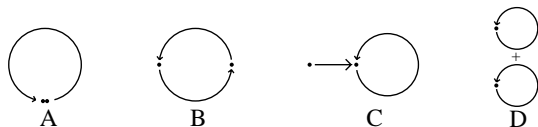
There is a one-to-one correspondence between ideal points and escape regions.

In \mathcal{S}_p itself, each escape region \mathcal{E} is a **punctured** disc.

Thus, in each escape region, one can define **equipotentials** and **external rays**. These provide a powerful method for studying the dynamics for maps $f \in \partial\mathcal{E}$.

Hyperbolic Components

A rational map is called **hyperbolic** if every critical orbit converges to an attracting or superattracting cycle.

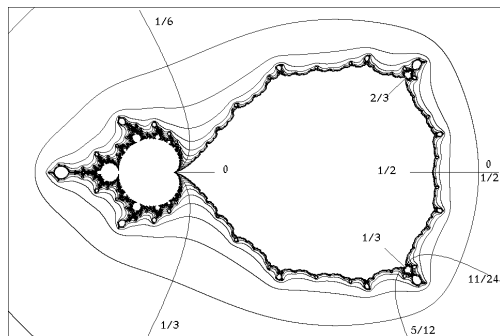


There are 4 types of hyperbolic components in $\mathcal{C}(S_p)$, indicated schematically above.

- A.** Adjacent critical points: in the same Fatou component.
- B.** Bicritical: in the same cycle of Fatou components.
- C.** Capture of one critical orbit by the Fatou cycle of the other.
- D.** Disjoint cycles of Fatou components. (Each Type D component in S_p is contained in a copy of the Mandelbrot set.)

Example: Period 1

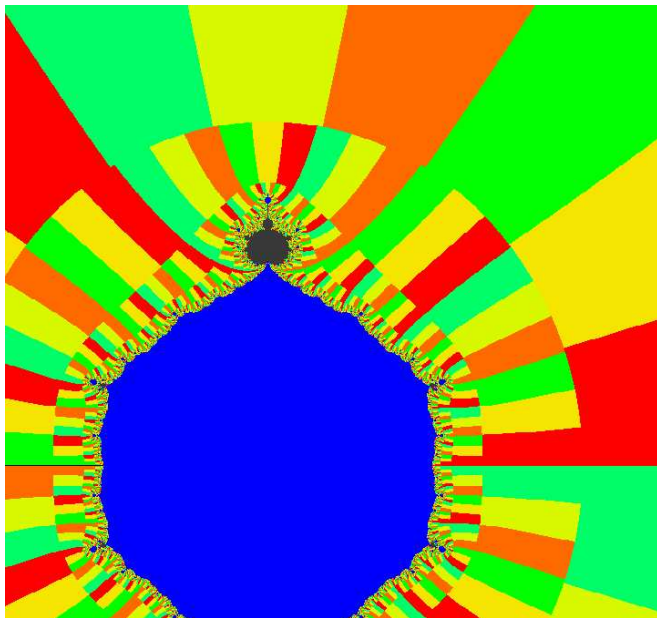
The curves \mathcal{S}_1 and $\mathcal{S}_1/\mathcal{I}$ are conformally isomorphic to \mathbb{C} ,
with one puncture point (at infinity) and one escape region.



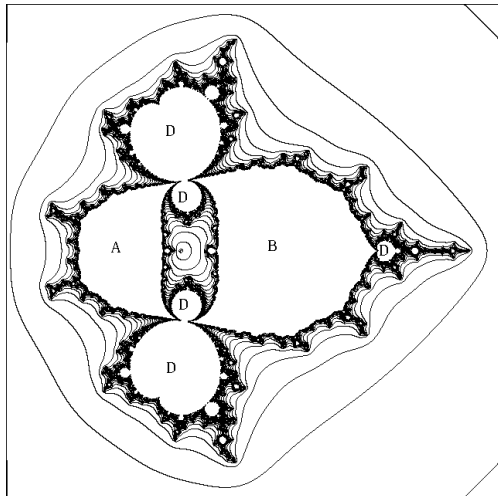
Bifurcation locus in $\mathcal{S}_1/\mathcal{I}$.

The 2-fold branched covering space \mathcal{S}_1 is branched over the
“center” point ($f_{0,0}(z) = z^3$) of the large component.

A Picture of \mathcal{S}_1



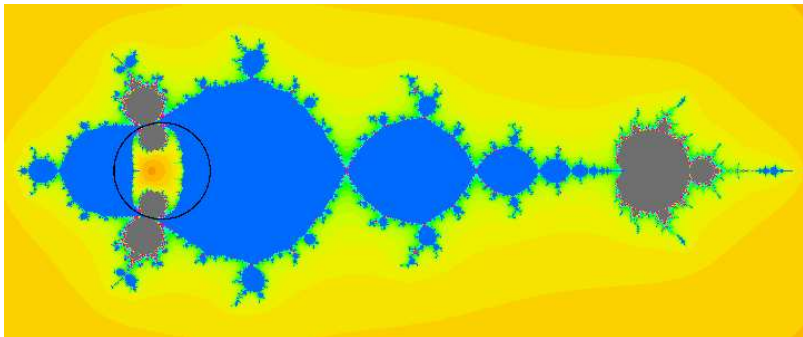
Period 2



The curve S_2/I is isomorphic to $\mathbb{C} \setminus \{0\}$, with two puncture points (at zero and infinity), and two escape regions.

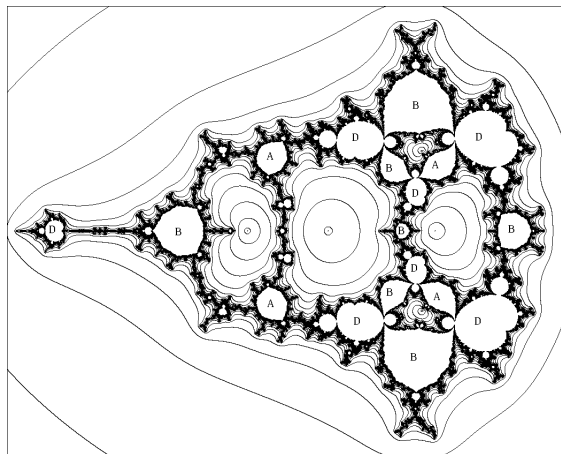
The two-fold covering space S_2 is branched over these two puncture points.

Another view of $\mathcal{S}_2/\mathcal{I}$



Here the inner and outer escape regions have been interchanged by inversion in the black circle.

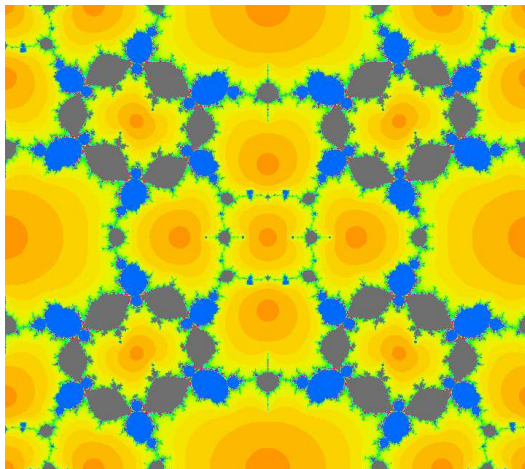
Period 3



The curve $\mathcal{S}_3/\mathcal{I}$ has genus zero, with six puncture points, hence six escape regions.

The covering space \mathcal{S}_3

\mathcal{S}_3 is a two-fold covering of $\mathcal{S}_3/\mathcal{I}$, branched over four of its six puncture points. Hence \mathcal{S}_3 has genus one, with eight punctures.



View of the universal covering space of this torus \mathcal{S}_3 .

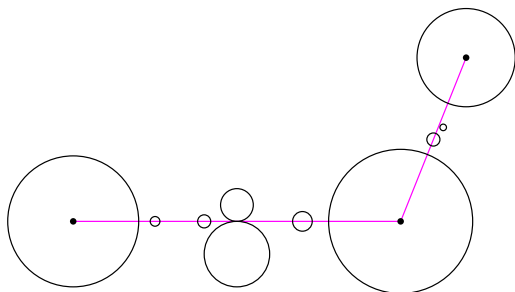
Boundaries of Hyperbolic Components

Assertion. *Every hyperbolic component H in $\mathcal{C}(S_p)$ is conformally an open disk with a preferred center point.*

Conjecturally, it is bounded by a simple closed curve.

(Pascale Roesch and Yin Yongcheng; work in progress.)
In the period one case, this was proved by Darroch Faught (1992), and by Roesch (1999, 2006).

Regulated Paths in the Connectedness Locus



DEFINITION. A path in $\mathcal{C}(\mathcal{S}_p)$ is **regulated** if its intersection with the closure \overline{H} of each hyperbolic component H is either:

- a single point or \emptyset ,
- a Poincaré geodesic joining a boundary point to the center, or
- a broken geodesic joining one boundary point to another via the center.

Regulated Paths and Curves

PROBLEM: Can any two centers be joined by at least one regulated path?

(In particular, is \mathcal{S}_p connected?)

We can also consider simple closed curves $\Gamma \subset \mathcal{C}(\mathcal{S}_p)$.

Definition. Such a curve is **regulated** if

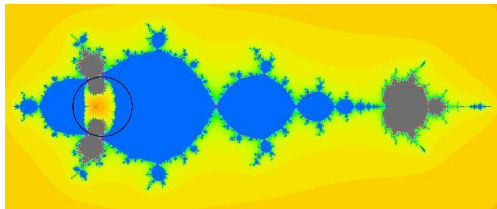
- it satisfies the analogous restrictions on $\Gamma \cap \overline{H}$ (but with no end points allowed), and if
- it contains at least one hyperbolic point. (This second condition is hopefully redundant.)

Assertion: A simple closed regulated curve in $\mathcal{C}(\mathcal{S}_p)$ cannot be homotopic to a point within \mathcal{S}_p .

A Conjectural Description of $\overline{\mathcal{S}}_p$

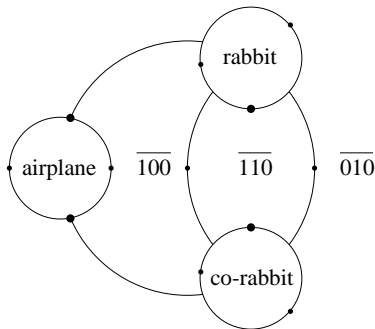
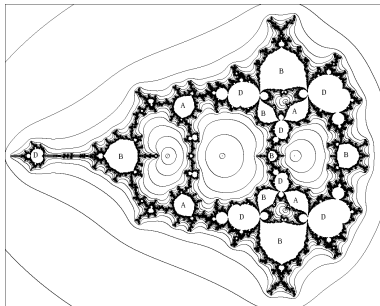
The claim is that there is a canonical cell subdivision of $\overline{\mathcal{S}}_p$ (or of $\overline{\mathcal{S}}_p/\mathcal{I}$). For $p > 1$ it can be described as follows:

- The 1-skeleton of this cell subdivision is the union of all simple closed regulated curves in the connectedness locus.
- The complement of the 1-skeleton in $\overline{\mathcal{S}}_p$ or $\overline{\mathcal{S}}_p/\mathcal{I}$ is a disjoint union of open 2-cells, one centered at each ideal point, and hence one 2-cell containing each escape region.



Example: For $\overline{\mathcal{S}}_2/\mathcal{I}$ there is only one simple closed regulated curve, shown in black. It separates the 2-sphere into two 2-cells, each containing one of the two escape regions.

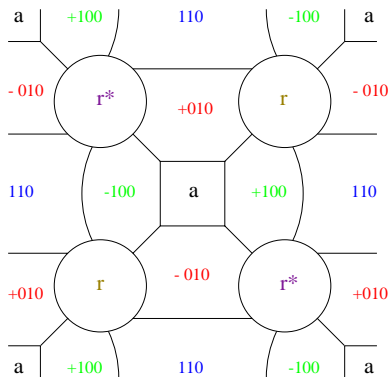
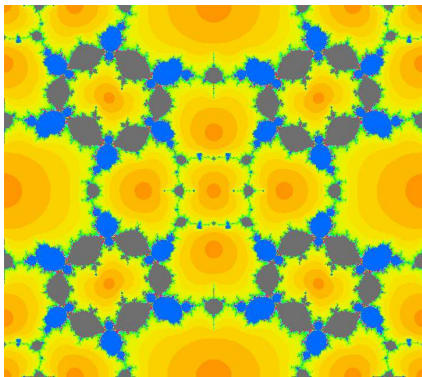
Example S_3/\mathcal{I} :



showing a cartoon of the cell structure on the right.

To describe these cell structures, it is essential to have some way to label the various escape regions!

Example \mathcal{S}_3 :



Corresponding pictures for the 2-fold covering \mathcal{S}_3 (lifted to its universal covering plane).

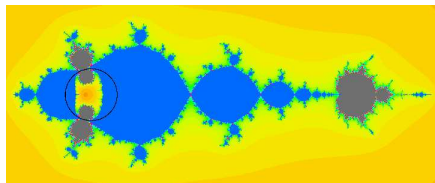
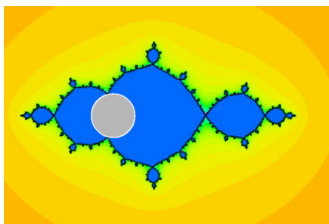
The involution \mathcal{I} corresponds to an 180° rotation of either of these figures.

Embedding $K(q)$ in $\mathcal{S}_p/\mathcal{I}$

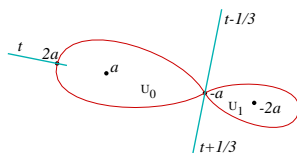
CONJECTURAL DESCRIPTION:

Each critically periodic $q(z) = z^2 + c$ of period p determines a corresponding 2-cell \mathbf{e}_q in $\mathcal{S}_p/\mathcal{I}$.

The filled Julia set $K(q)$, cut open along its minimal Hubbard tree, embeds canonically in \mathbf{e}_q , with the cut open tree mapping to $\partial\mathbf{e}_q$.



The Kneading Sequence of an Escape Region.



Suppose that the orbit of $+a$ under the map $f = f_{a,v}$ is bounded, but the orbit of $-a$ escapes to infinity.

Then the equipotential through $-a$ is a figure eight curve.

Let U_0 and U_1 be the bounded complementary components, with $a \in U_0$. Any bounded orbit $z = z_1 \mapsto z_2 \mapsto \dots$ determines an infinite sequence $\vec{\sigma}(z) = (\sigma_1, \sigma_2, \dots)$ of zeros and ones, with

$$z_j \in U_{\sigma_j}.$$

Definition. The sequence $\vec{\sigma}(v)$ associated with the critical value $v = f(a)$ will be called the **kneading sequence** $\vec{\sigma}_f$.

The Associated Quadratic Map

Now suppose that the critical point a is periodic of period p . In other words, suppose that $f = f_{a,v} \in \mathcal{S}_p$.

Evidently the kneading sequence $\vec{\sigma}_f$ is also periodic, and the period of $\vec{\sigma}_f$ must be some divisor d of the period p of f . In particular, $\sigma_d = \sigma_p = 0$.

A convenient notation: Set $\vec{\sigma}_f = \overline{\sigma_1 \sigma_2 \cdots \sigma_{p-1} 0}$.

Branner and Hubbard (1992): *For each such map f , there is a critically periodic quadratic polynomial $q(z) = z^2 + c$ with critical period p/d , such that every nontrivial component of the cubic Julia set $J(f)$ is a copy of the quadratic Julia set $J(q)$.*

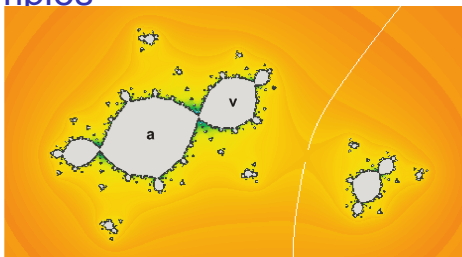
Primitive Escape Regions

Example. The quadratic polynomial $q(z)$ has critical period $p/d = 1$ if and only if $q(z) = z^2$, with a circle as Julia set.

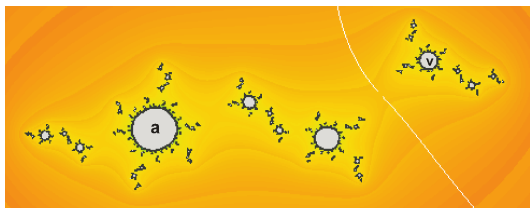
Corollary: For $f \in \mathcal{E} \subset \mathcal{S}_p$, each non-trivial component of $J(f)$ is a topological circle if and only if the kneading sequence $\vec{\sigma}_f$ has period d exactly equal to p .

This case $p = d$ will be called the **primitive** case.

Period 2 Examples



Here the kneading sequence is $\overline{00}$, and the associated quadratic map is $z^2 - 1$.



Here the kneading sequence is $\overline{10}$ (primitive case).

Multiplicity

Define the **multiplicity** μ of an escape region $\mathcal{E} \subset \mathcal{S}_\rho$ to be the number of intersections of \mathcal{E} with a line of the form

$$\{(a, v) \in \mathbb{C}^2 \ ; \ a = \text{large constant}\}.$$

Then the number of escape regions, counted with multiplicity, is equal to the degree of the affine curve \mathcal{S}_ρ .

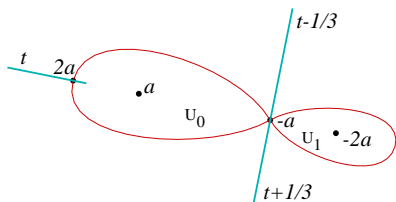
Theorem. *For $|a|$ large, the escape region \mathcal{E} can be parametrized by $\sqrt[\mu]{a}$, where μ is its multiplicity.*

In particular, every point $a_j = f^{\circ j}(a)$ of the critical orbit can be expressed as a holomorphic function of $\sqrt[\mu]{a}$.

A Change of Variable

As $|a| \rightarrow \infty$, we have the asymptotic estimate

$$a_j = \begin{cases} a + O(1) & \text{if } \sigma_j = 0 \\ -2a + O(1) & \text{if } \sigma_j = 1. \end{cases}$$



It will be convenient to replace z by the new variable $s(z) = (a - z)/3a$, with $s(a) = 0$ and $s(-2a) = 1$. In terms of this variable s , every point $s_j = s(a_j)$ on the critical orbit is very close to either $s = 0$ or $s = 1$:

$$s_j = \sigma_j + O(1/a) \quad \text{as} \quad |a| \rightarrow \infty.$$

Puiseux Series

(See Kiwi, 2006 for a closely related exposition.)

It is convenient to set $t = 1/3a$, and to use $t^{1/\mu}$ as parameter for $\mathcal{E}^+ = \mathcal{E} \cup (\text{ideal point}) \subset \overline{\mathcal{S}}_p$ near the ideal point $t = 0$.

Then we can think of s_j as a holomorphic function of $t^{1/\mu}$ for $|t|$ small, with $s_j(0) = \sigma_j \in \{0, 1\}$.

Alternatively we can think of s_j as a power series in $\mathbb{C}[[t^{1/\mu}]]$.

Let \widehat{s}_j be the first non-zero term in this power series.

Assertion. For periods $p \leq 4$, the power series s_1, \dots, s_{p-1} are uniquely determined by the $p - 1$ monomials $\widehat{s}_1, \dots, \widehat{s}_{p-1}$. Furthermore, if we write these monomials as $\widehat{s}_j = k_j t^{n_j/\mu}$, then each coefficient k_j is an algebraic unit.

Question: Are these statements still true for $p > 4$?

The “Easy” Case

Notation: If $\widehat{s}_j = k_j t^{n_j/\mu}$, set $\text{ord}(s_j) = n_j/\mu \geq 0$.

Suppose now that s_1, \dots, s_{p-1} satisfy the condition that $\text{ord}(s_j) < 2$.

(For periods $p \leq 4$, this condition is satisfied if and only if the kneading sequence is primitive.)

ASSERTION. In this easy case, there is a strongly convergent algorithm for computing the s_j from the \widehat{s}_j .

Futhermore the coefficient k_j of each monomial \widehat{s}_j is a root of unity, $k_j^{2^{p-1}} = 1$,

and all of the coefficients for the series s_j belong to the ring generated over $\mathbb{Z}[1/2]$ by these roots of unity.

Example: The Period Two Case

For $p = 2$ there is only one primitive kneading sequence $\vec{\sigma}_f = \overline{10}$, hence $\widehat{s}_1 = 1$, and $\text{ord}(s_1) = 0$.

In this case, the algorithm reduces to iteration of

$$s_1 \mapsto 1 - t^2/s_1 \quad \text{starting with } s_1 = 1.$$

This converges rapidly to

$$s_1 = \frac{1}{2} \left(1 + \sqrt{1 - 4t^2} \right) = 1 - t^2 - t^4 - 2t^6 - \dots$$

$$\in \mathbb{Z}[[t]].$$

Equations to Solve

We want

$$a_{j+1} = f(a_j) \quad \text{for } 1 \leq j < p, \quad \text{with } a_p = a.$$

Equivalently

$$a_{j+1} - a_1 = (a_j - a)^2(a_j + 2a).$$

or

$$t^2(s_{j+1} - s_1) = s_j^2(s_j - 1).$$

Algorithm: Map (s_1, \dots, s_{p-1}) to (s'_1, \dots, s'_{p-1}) ,
where

$$s'_j = 1 + t^2(s_{j+1} - s_1)/s_j^2 \quad \text{if } \sigma_j = 1,$$

$$s'_j = \pm \sqrt{t^2(s_{j+1} - s_1)/(s_j - 1)} = s_j \sqrt{(t/s_j)^2(s_{j+1} - s_1)/(s_j - 1)}$$

if $\sigma_j = 0$.