# The Relative Green's Function

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work with

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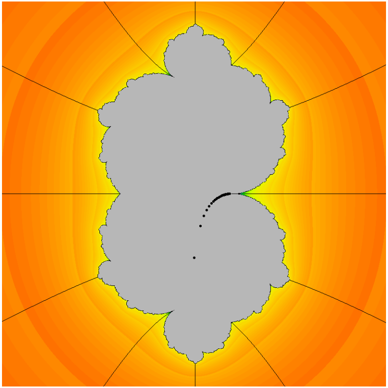
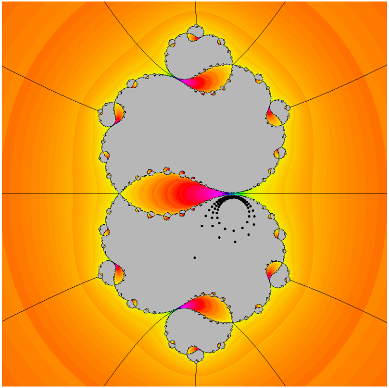
# An Example





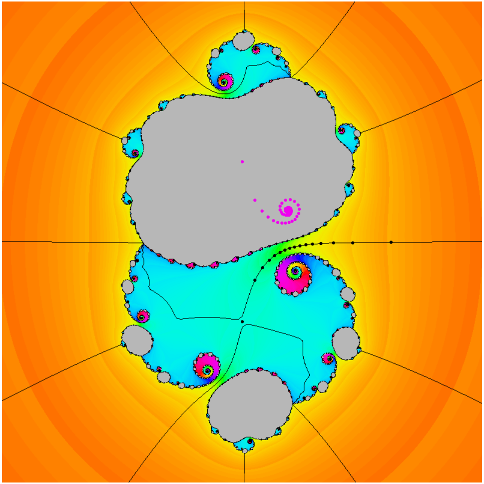
$F(z) = z^3 + .75z + .04811$

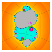


$F(z) = z^3 + .75z + .055$

# A Non-Real Approximation







$$F(z) = z^3 + .75\,z + (.08 + .0089\,i)$$

# The Green's Function: Three Versions.



**(1) In the $z$-plane.** For any polynomial function $F$ of degree $d \geq 2$,

$$\mathbf{g}_F(z) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ |F^{\circ n}(z)| \geq 0 .$$

Then

- $\mathbf{g}_F\big(F(z)\big) = d \cdot \mathbf{g}_F(z)$ ,
- $\mathbf{g}_F(z) = 0 \iff z \in K(F)$ , *and*
- $\mathbf{g}_F$ *is continuous everywhere and harmonic throughout* $\mathbb{C} \smallsetminus K(F)$.

**(2) In parameter space.** Define $\mathbf{G}(F) = \max_{F'(c)=0} \mathbf{g}_F(c)$ .

**(3) The relative Green's function.** If $\mathbf{G}(F) > 0$, set

$$\mathbf{rg}_F(z) = \mathbf{g}_F(z)/\mathbf{G}(F) .$$

In practice we will assume that there is a ***marked critical point*** $\mathbf{c}$ with $\mathbf{g}_F(\mathbf{c}) = \mathbf{G}(F)$; so that $\mathbf{rg}_F(z) = \mathbf{g}_F(z)/\mathbf{g}_F(\mathbf{c})$.

# External rays in the $z$-plane. 

Now assume that $F$ is ***monic***, so that

$$F(z) \sim z^d \qquad \text{as} \quad |z| \to \infty .$$

The orthogonal trajectories to the family of equipotentials $\mathbf{g}_F(z) = \text{constant}$ are called ***dynamic rays,*** denoted by $\mathcal{R}_F(\theta)$ where $\theta \in \mathbb{R}/\mathbb{Z}$ is the angle, measured at infinity.

Every such ray either ***terminates*** when it hits a critical or pre-critical point of $F$, or else accumulates on $J(F)$.

Note that

$$F\big(\mathcal{R}_F(\theta)\big) \subset \mathcal{R}_F(d \cdot \theta) ,$$

where $d$ is the degree.

For example, $F$ always maps the zero-ray $\mathcal{R}_F(0)$ into itself.

# Theorem 1: Hypothesis.



Let $\{F_j\}$ be a sequence of monic polynomial maps of degree $d$, with $\mathbf{G}(F_j) \searrow 0$ as $j \to \infty$.

Suppose that each $F_j$ has a marked critical point $\mathbf{c}_j$ with $\mathbf{g}_j(\mathbf{c}_j) = \mathbf{G}(F_j)$.

Suppose that each marked critical value $\mathbf{v}_j = F_j(\mathbf{c}_j)$ belongs to the dynamic ray $\mathcal{R}_{F_j}(\theta)$, for some fixed angle $\theta \in \mathbb{Q}/\mathbb{Z}$.

Finally, suppose that

$$\lim F_j = F \qquad \text{and} \qquad \lim \mathbf{c}_j = \mathbf{c} \; ,$$

where $\mathbf{c}$ belongs to a cycle of parabolic basins for $F$.

Let $\mathcal{B}$ be the ***total parabolic basin*** consisting of all points whose orbit under $F$ enters this cycle.

# Theorem 1: Conclusion.



After passing to a suitable infinite subsequence of $\{F_j\}$,

> *the relative Green's functions* $\mathbf{rg}_{F_j}$ *converge locally uniformly throughout* $\mathcal{B}$ *to a continuous function* $\mathbf{rg}(z) \geq 0$ *which is harmonic on the open subset* $\mathcal{B}^*$ *where* $\mathbf{rg}(z) > 0$.

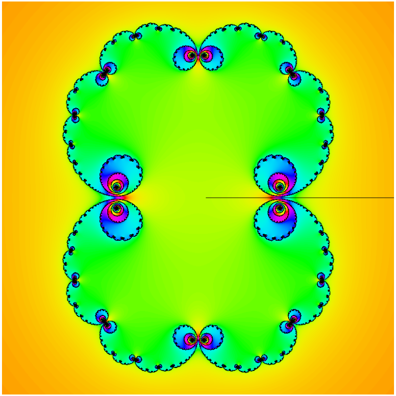
Furthermore

$$\mathbf{rg}\big(F(z)\big) \;=\; d \cdot \mathbf{rg}(z)\;.$$

(In fact **rg** restricted to $\mathcal{B}^*$ is the real part of a holomorphic function from $\mathcal{B}^*$ to the right-half plane $\{u + iv\;;\; u > 0\}$ which satisfies the corresponding identity.)

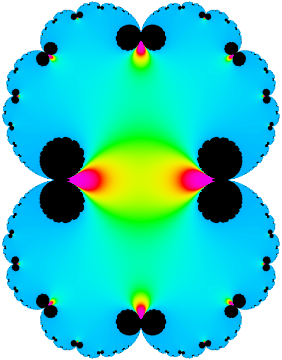
# Example: The Cauliflower Map $F(z) = z^2 + z$





Julia set for

$$z \mapsto z^2 + z + .004$$



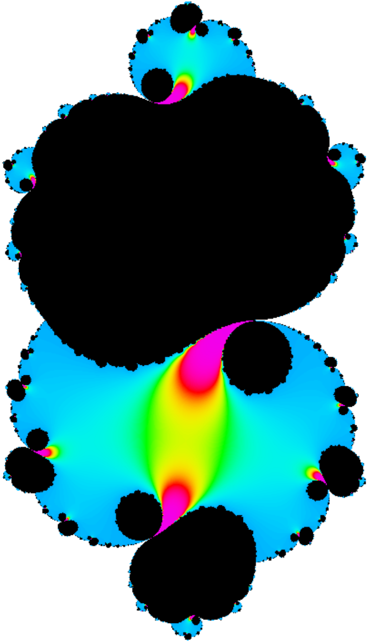
A limiting relative Green's function for

$$z \mapsto z^2 + z$$

# Our First Example



Limiting relative Green's function for a parameter ray landing on $F(z) = x^3 + .75\,z + .04811$.



# Notations for the proof.



For any monic $f(z)$ of degree $d \geq 2$, and any constant $g \geq \mathbf{G}(f)$, let

$$\Omega_g(f) \subset \mathbb{C}$$

be the neighborhood of infinity consisting of all $z$ with $\mathbf{g}_f(z) > g$.

Since there are no critical points in $\Omega_g(f)$, there is a Böttcher isomorphism $\mathfrak{b}_f : \Omega_g(f) \xrightarrow{\cong} \mathbb{C} \smallsetminus \overline{\mathbb{D}}_{\exp(g)}$. The universal covering space $\widetilde{\Omega}_g(f)$ can be identified with the right half-plane $\mathbb{H}_g = \{u + iv\ ;\ u > g\}$, with projection map $\mathfrak{p} : \mathbb{H}_g \to \Omega_g(f)$ given by

$$\mathbb{H}_g \xrightarrow{\exp} \mathbb{C} \smallsetminus \overline{\mathbb{D}}_{\exp(g)} \xrightarrow{\mathfrak{b}_f^{-1}} \Omega_g(f)\ .$$

Note that $\mathfrak{p}$ sends the real axis in $\mathbb{H}_g$ onto the zero dynamic ray in $\Omega_g$ .

Note also that $f : \Omega_g(f) \xrightarrow{\cong} \Omega_{d\cdot g}$ lifts to the linear map $w \mapsto d \cdot w$ from $\mathbb{H}_g$ to $\mathbb{H}_{d\cdot g}$ .

# Understanding $(f^{-1})^{\circ n}$.

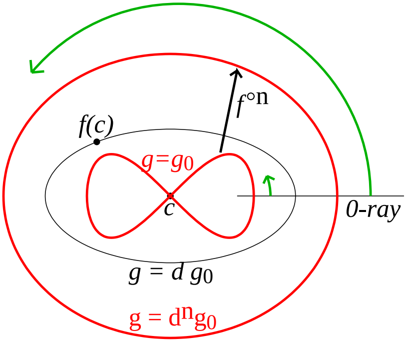
Let $f$ be monic of degree $d$ and let $g_0 \geq \mathbf{G}(f)$.

**Main Lemma.**
For any $n \geq 1$ there is a commutative diagram of holomorphic maps

$$\begin{array}{ccc} \mathbb{H}_{g_0} & \xrightarrow[\cong]{\cdot d^n} & \mathbb{H}_{d^n \cdot g_0} \\ \downarrow \mathfrak{p} & \swarrow \psi & \downarrow \mathfrak{p} \\ \Omega_{g_0} & \xrightarrow{f^{\circ n}} & \Omega_{d^n \cdot g_0} \end{array},$$

where $\psi(d \cdot w) = f(\psi(w))$, and

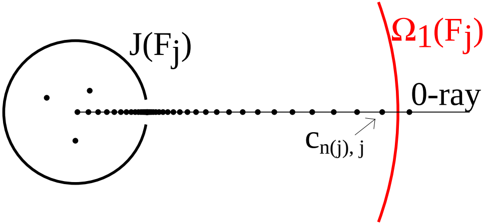
$$\mathbf{g}_f(\psi(w)) = \Re(w)/d^n.$$



Remember that each $\mathbf{v}_j = F_j(\mathbf{c}_j)$ belongs to the $\theta$-ray $\mathcal{R}_{F_j}(\theta)$. Since $\theta$ eventually maps to zero under multiplication by $d$, for each $F_j$, **most** points of the critical orbit

$$F_j : \mathbf{c}_j = \mathbf{c}_{0,j} \mapsto \mathbf{c}_{1,j} \mapsto \mathbf{c}_{2,j} \mapsto \cdots$$

must belong to the zero ray $\mathcal{R}_{F_j}(0)$.




Let $\mathbf{c}_{n(j),j}$ be the last orbit point with $\mathbf{g}_{F_j}(c_{n(j),j}) < 1$.

Then $\psi_j : \mathbb{H}_1 \to \Omega_{1/d^{n(j)}}$ maps $\mathbb{R} \cap \mathbb{H}_1$ to the zero ray, with

$F_j(\psi_j(u)) = \psi_j(d \cdot u)$ and $\mathbf{g}_{F_j}(\psi_j(u)) = u/d^{n(j)}$.

Let $K$ be any compact subset of $\mathbb{H}_1$.

*The successive images $\psi_j(K) \subset \mathbb{C}$ have uniformly bounded Green's function, hence are uniformly bounded.*

Thus by Montel's Theorem, we can choose a locally convergent subsequence of $\{\psi_j|_{\mathrm{interior}(K)}\}$.

*Repeating this for larger and larger $K$, we can find a subsequence which converges locally uniformly to a holomorphic map $\mathbf{\Psi} : \mathbb{H}_1 \to \mathbb{C}$.*

**Lemma.** The image $\mathbf{\Psi}(\mathbb{H}_1)$ is an open subset of $K(f) \smallsetminus J(f)$ which contains all but finitely many points of the orbit of **c.**

**Proof Outline.** The map $\mathbf{\Psi}$ is not constant since the images of points on the critical orbit are distinct. Hence it is univalent by a theorem of Hurwitz. The image $U_0 = \mathbf{\Psi}(\mathbb{H}_1)$ is open, $F$-invariant, and bounded.

Hence it can't intersect the Julia set. □

# A Holomorphic Relative Green's Function



Thus we have a conformal isomorphism

$$\boldsymbol{\Psi} : \mathbb{H}_1 \xrightarrow{\cong} U_0 \subset U \subset \mathcal{B}^*$$

with $\boldsymbol{\Psi}(d \cdot w) = F\big(\boldsymbol{\Psi}(w)\big)$. Hence the inverse isomorphism

$$\boldsymbol{\Psi}^{-1} : U_0 \xrightarrow{\cong} \mathbb{H}_1 \ .$$

satisfies $\qquad \boldsymbol{\Psi}^{-1}\big(F(z)\big) \ = \ d \cdot \boldsymbol{\Psi}^{-1}(z) \ .$

> **Lemma.** $\boldsymbol{\Psi}^{-1}$ *extends uniquely to a holomorphic map $\mathcal{G}$ from $\mathcal{B}^*$ to the right half-plane $\mathbb{H}_0$ satisfying the corresponding identity* $\mathcal{G}\big(F(z)\big) = d \cdot \mathcal{G}(z)$ . *Furthermore the real part $\Re\big(\mathcal{G}(z)\big)$ coincides with the limiting relative Green's function $\mathbf{rg}(z) = \lim_{j\to\infty} \mathbf{rg}_j(z)$ up to a multiplicative constant.*

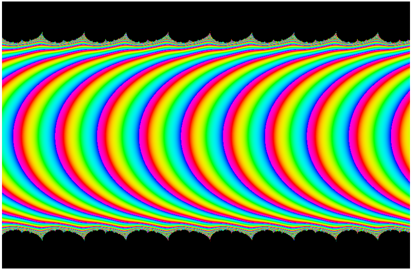
(The precise formula is $\quad \mathbf{rg}(z) = \Re\big(\mathcal{G}(z)\big)/g_c \quad$ where $g_c = \lim_{j\to\infty} \mathbf{g}_j(\mathbf{c}_j)\, d^{n(j)}$.)

The **Fatou coordinate** on $\mathcal{B}$ is the unique holomorphic map

$$\Phi : \mathcal{B} \to \mathbb{C} \qquad \text{such that}$$

**(1)** $\Phi\big(F(z)\big) = \Phi(z) + 1$ , and
**(2)** $\Phi(\mathbf{c}) = 0$ .

Two points of $\mathcal{B}$ are ***eventually equal*** under $F$, that is $F^{\circ n}(z) = F^{\circ n}(z')$ for some $n$, if and only if $\Phi(z) = \Phi(z')$.
It follows easily that $\mathbf{rg}(z)$ is uniquely determined by $\Phi(z)$.



Plot of $\log_2\big(\mathbf{rg}(z)\big)$ in the $\Phi(z)$ plane for

$$F(z) = z^2 + z \, .$$

**Theorem 2.** The quotient of $\Phi(\mathcal{B}^*)$ under unit translation is an annulus of modulus $\pi/\log(d^q)$.

# The Limiting $\theta$-Ray.

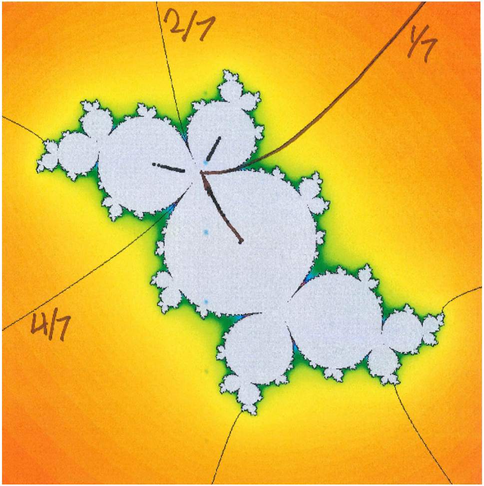


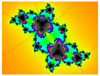
Recall that each dynamic ray $\mathcal{R}_{F_j}(\theta)$ passes through the marked critical value $F_j(\mathbf{c}_j)$. Since $\theta$ is rational, it is eventually periodic.

**Theorem 3.** For each $k \geq 0$, the sequence of rays $\mathcal{R}_{F_j}(d^k\theta)$ converges locally uniformly as $j \to \infty$ to a ***limit ray*** $\mathbf{R}(d^k\theta)$, which is smooth except at a single point where it crosses the Julia set $J(F)$, passing from the basin of infinity to the Fatou component containing $F^{\circ k+1}(\mathbf{c})$. Within $\mathcal{B}$, this limit ray is an orthogonal trajectory to the family of equipotentials $\mathbf{rg}(z) = \mathrm{constant}$. This limit ray extends until it either terminates at a critical or pre-critical point of $F$, or until it accumulates on the boundary of the locus $\mathbf{rg}(z) = 0$.

# An Example: The Fat Rabbit





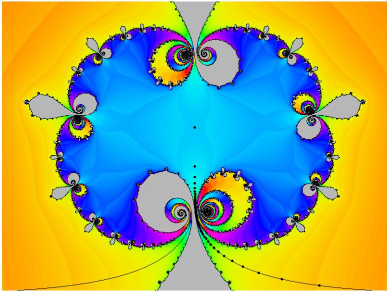



$$F(z) = z^2 + e^{2\pi i/3} z\ .$$

A limit of maps where the critical orbit escapes along the $1/7$ ray.

# Another Example





$$F(z) \simeq z^3 + i\,z^2 + z$$

# With an Extra Hypothesis.



**Theorem 4.**

Now suppose that **c** is the only critical point in $\mathcal{B}^*$. Then:

**(1)** The angle $\theta$ is strictly periodic, say of period $q$.

**(2)** Each limit ray $\mathbf{R}(d^k\theta)$ terminates at the unique critical point of $F^{\circ q}$ in the basin of $F^{\circ k+1}(\mathbf{c})$

**(3)** The intersection of each component of $\mathcal{B}$ with $\mathcal{B}^*$ is connected and simply-connected.

QUESTION: Are these statements true without the extra hypothesis?

# The Lavaurs-Oudkerk Limit.



**Conjecture.** If our maps $F_j$ belong to a parameter ray in a one complex dimensional space of polynomials, then there is a circle of possible limits **rg**. The limit is uniquely determined by the ***phase parameter***

$$\log_d(g_{\mathbf{c}}) = \lim_{j\to\infty}\Big(\log_d\left(\mathbf{g}_j(\mathbf{c}_j)\right) \pmod{\mathbb{Z}}\Big) \in \mathbb{R}/\mathbb{Z}\,.$$

- A. Bonifant, J. Milnor and S. Sutherland, *Parabolic Implosion and the Relative Green's Function*, manuscript in progress.
- R. Oudkerk, *The Parabolic Implosion: Lavaurs Maps and Strong Convergence for Rational Maps*, Contemporary Mathematics **303** (2002) 79–105.
- C. Petersen and G. Ryd, *Convergence of rational rays in parameter spaces*, "The Mandelbrot Set, Theme and Variations," 161–172, London Math. Soc. Lecture Note Ser. **274**, Cambridge Univ. Press, 2000.