Cubic Maps and the Mandelbrot Set

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Definitions

Let $\mathcal{S}_p$ be the space of all monic centered cubic polynomial maps F with a **marked critical point** of period $p \geq 1$.
Setting $F(z) = z^3 - 3a^2z + b$, the critical points are $\pm a$.
Here $+a$ will always be the marked critical point.
If $v = F(a)$ is the corresponding marked critical value, we can solve for $b = 2a^3 + v$.

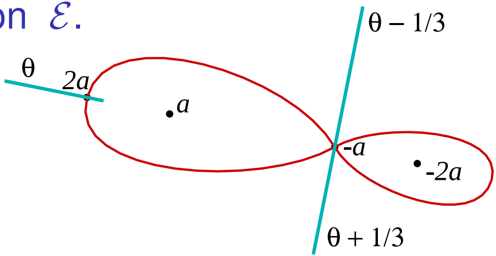
Identify $\mathcal{S}_p$ with the smooth affine curve consisting of all pairs $(a, v) \in \mathbb{C}^2$ such that a has period exactly p under iteration of F.

Theorem of Arfeux and Kiwi: Every $\mathcal{S}_p$ is connected.

p :	1	2	3	4	5	6
genus	0	0	1	15	93	393
# punctures	1	2	8	20	56	144

The **connectedness locus**, consisting of all $F \in \mathcal{S}_p$ such that $J(F)$ is connected, is a compact and connected subset of $\mathcal{S}_p$.
Its complement consists of finitely many **escape regions**, each biholomorphic to $\mathbb{C} \smallsetminus \overline{\mathbb{D}}$.

Cartoon of the dynamic plane for a map F in any escape region $\mathcal{E}$.

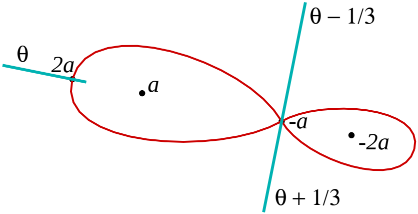


By definition, θ is the **parameter angle** for the parameter ray which passes through this map F.

Definition. *If either $\theta + 1/3$ or $\theta - 1/3$ has period q under tripling, then we will say that θ is* ***co-periodic*** *of* ***co-period*** *q.*

Theorem. A parameter ray of angle θ lands at a parabolic map if and only if θ is co-periodic. The cycle of parabolic basins has period q if and only if θ has co-period q.

The Kneading Invariant

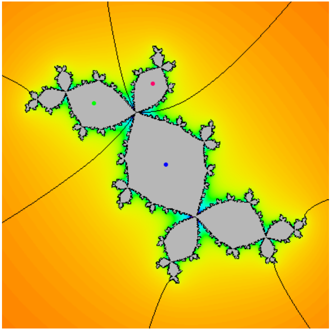


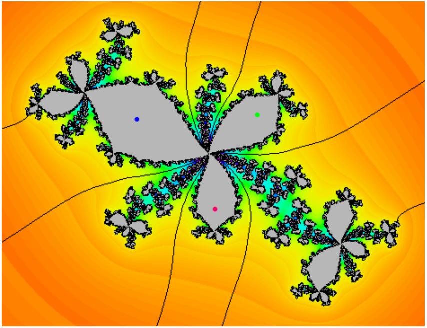
The **kneading invariant** $(i_1, \cdots, i_{p-1}, 0)$ *of* $\mathcal{E}$ *describes the way in which the orbit of* a *bounces back and forth between the two lobes of the figure eight.*

In particular, the kneading invariant is $(0, 0, \cdots, 0)$ if and only if the entire orbit of a is contained in the left hand lobe.

The Mandelbrot set and Escape Regions of $\mathcal{S}_p$.

There is a one-to-one correspondence between period p hyperbolic components in the classical Mandelbrot set $\mathbb{M}$, and escape regions in $\mathcal{S}_p$ with trivial kneading invariant.

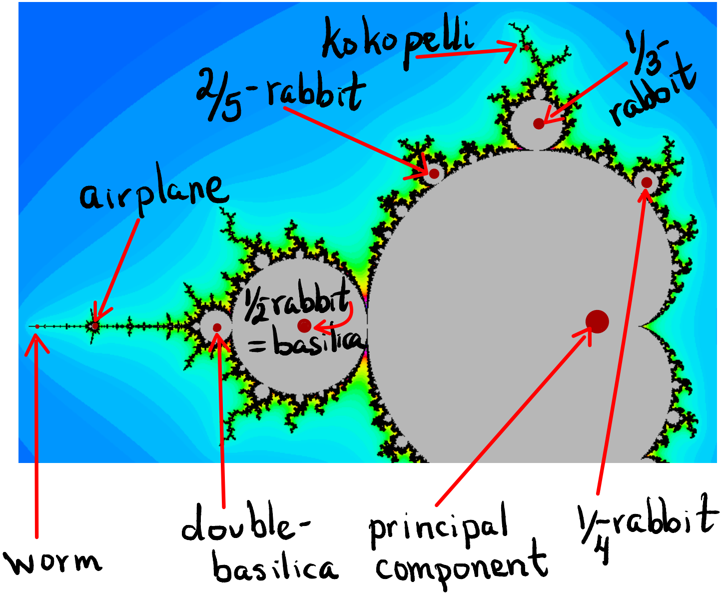




On the left: the Douady rabbit. On the right: a Julia set from the corresponding escape region in $\mathcal{S}_3$. Every non-trivial connected component is hybrid equivalent to the rabbit.

(The proof depends on the Branner-Hubbard puzzle.)

Some Hyperbolic components in $\mathbb{M}$.



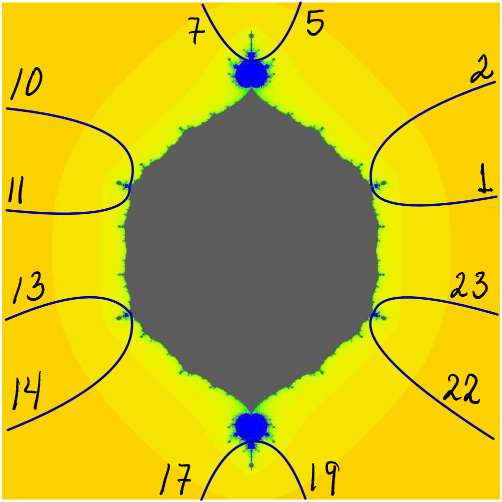
The Landing Together Conjecture.

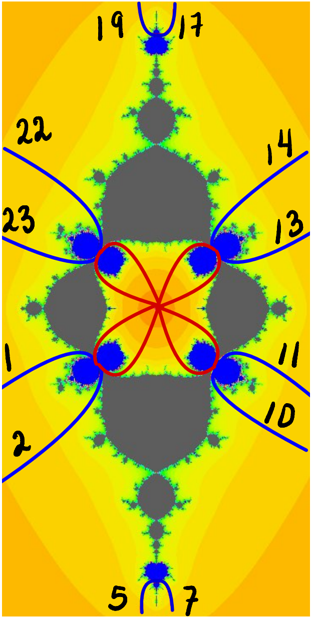
We will say that two parameter rays **land together** if they have the same landing point in $\mathcal{S}_p$.

Conjecture. In any zero-kneading region $\mathcal{E}$ and for any positive integer q, the parameter rays with angles of co-period q land together in pairs.
Furthermore if the rays of co-periodic angle θ and θ' land together in one zero-kneading region, then the corresponding rays land together in **every** zero-kneading region.

(In the special case of the zero-kneading regions in $\mathcal{S}_1$ and $\mathcal{S}_2$, the second part of this conjecture has been proved by Bonifant, Estabrooks and Sharland.)

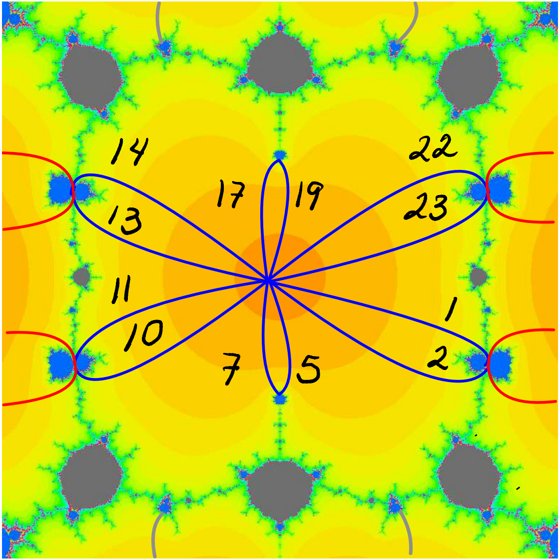
Example: Rays of Co-period 2 in $\mathcal{S}_1$ and $\mathcal{S}_2$.





Here the common denominator
is $3(3^q - 1) = 24$.

Example: The Airplane Region in $\mathcal{S}_3$.



Showing all rays of co-period two.

The Three or Four Conjecture.

In any zero-kneading region of $\mathcal{S}_p$, the parameter rays of co-period p play a very special role.

If two such rays in $\mathcal{E}$ land at a boundary point of $\mathcal{E}$ which is shared with one or more other escape regions, then we conjecture that there are **either one or two** *rays from outside of $\mathcal{E}$ which land at the same point,*

making a total of either three or four.

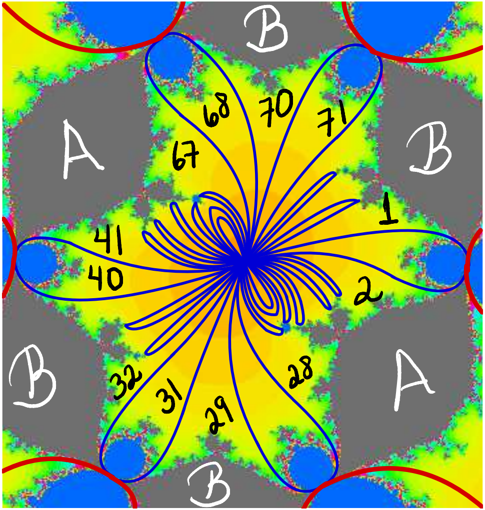
Hyperbolic components in $\mathbb{M}$ come in two types:

They are either **primitive** (with a cusp),

or a **satellite** (with no cusp).

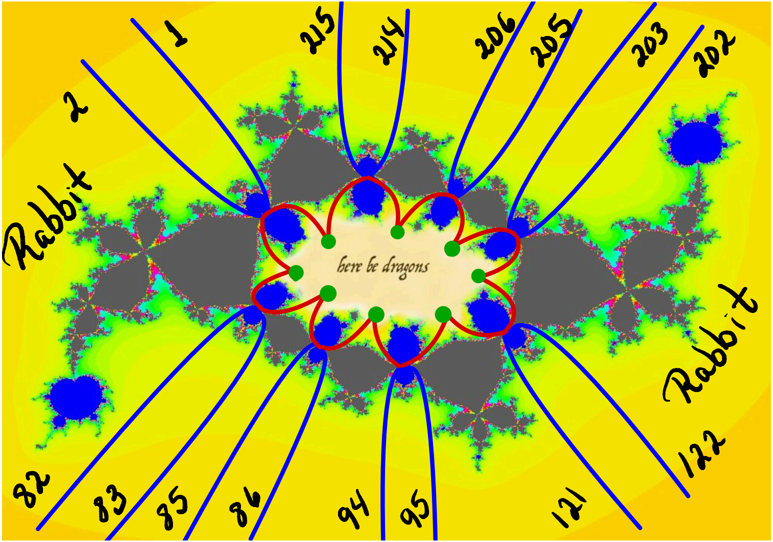
The Three or Four Conjecture (Satellite Case).

In this case, there are four rays landing at each shared boundary point, and $2p$ such boundary points.



Example: The $(1/3)$-rabbit region (denominator 78).

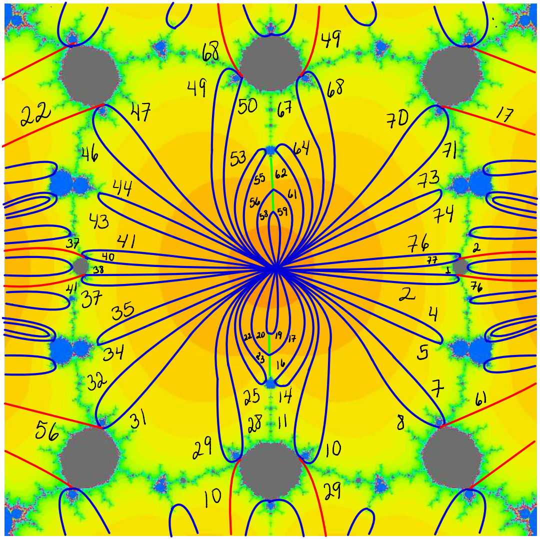
The $1/4$-Rabbit Region in $\mathcal{S}_4$



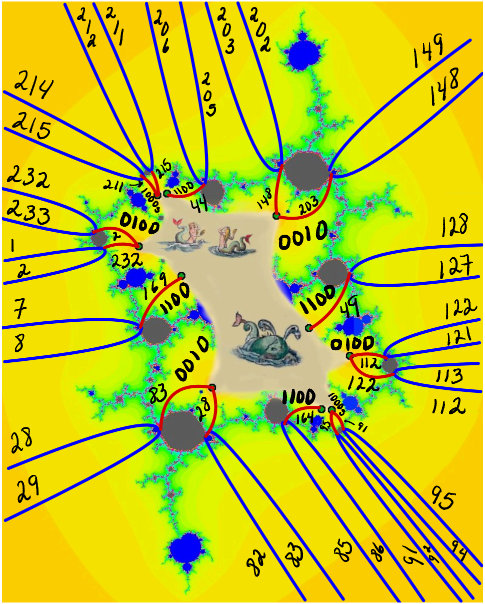
(with denominator 240)

The Three or Four Conjecture (Primitive Case).

In this case, there are three rays landing at each shared boundary point, and $4p$ such boundary points.



The Kokopelli Region in $\mathcal{S}_4$



Tessellations and Orbit Portraits.

For each $p \geq 1$ and each $q \geq 1$, the parameter rays of co-period q and their parabolic landing points divide the Riemann surface $\mathcal{S}_p$ into connected open sets which we call the **faces** of the **tessellation** $\mathrm{Tes}_q(\mathcal{S}_p)$.

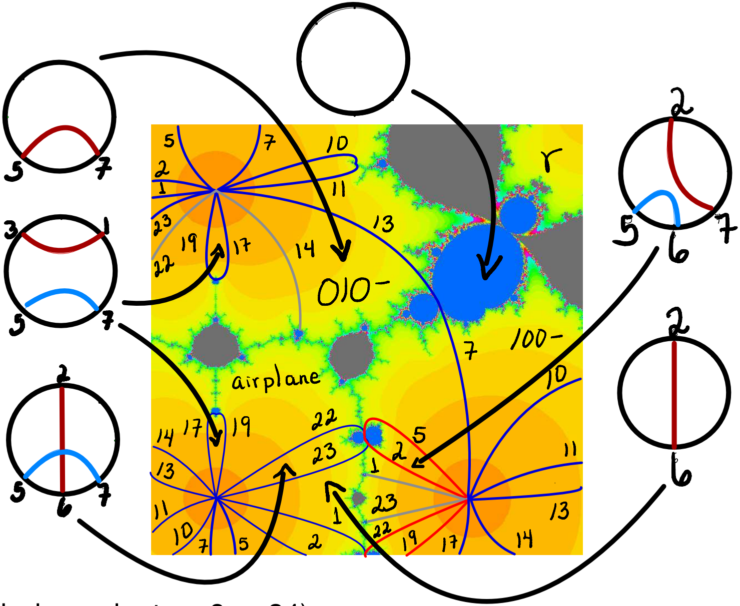
A basic invariant associated with each face is its ***period*** *q* ***orbit portrait****.*

Definition. The **orbit portrait** of a map F is the following equivalence relation between angles of period q under tripling:

Two angles θ and θ' are **equivalent** *if and only if the dynamic rays of angle θ and θ' for F land at a common point in the Julia set.*

Theorem. Two maps in the same face always have the same orbit portrait.

Example: Part of the tessellation $\mathrm{Tes}_2(\mathcal{S}_3)$.



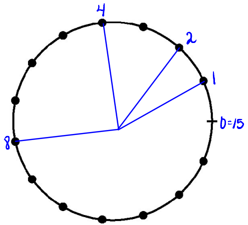
(with denominators 8 or 24)

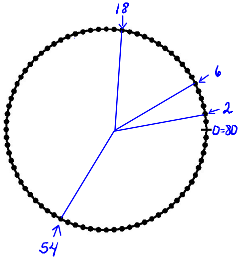
Quadratic to Cubic: Orbit Portrait Conjecture.

For any zero-kneading escape region $\mathcal{E} \subset \mathcal{S}_p$, we conjecture that there is a close relationship between:

(1) the orbit portrait for the root point of the associated Mandelbrot component, and

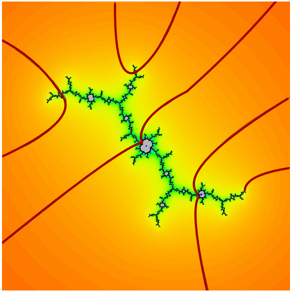
(2) the period p orbit portrait for **any one** of the shared faces around the boundary of $\mathcal{E}$.

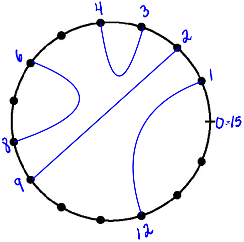




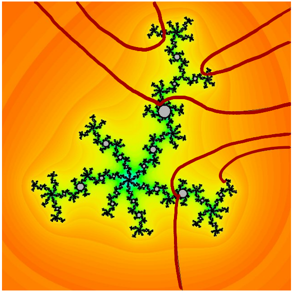
Left: Orbit Portrait for the root point of the (1/4)-rabbit in $\mathbb{M}$.
Right: Orbit portrait for one of the eight shared faces around the (1/4)-rabbit region in $\mathcal{S}_4$. (Denominators 15 and 80.)

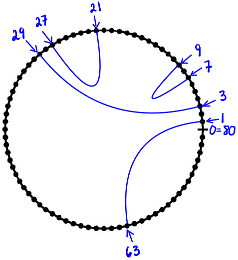
Another Example





The Kokopelli root point in $\mathbb{M}$.

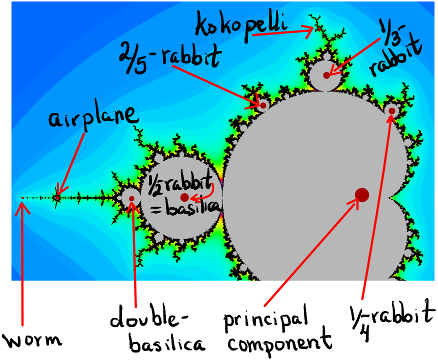




One of 16 shared faces around the Kokopelli region in $\mathcal{S}_4$.

The Mandelbrot Vein Conjecture.

By a **vein** in the Mandelbrot set we mean a connected path which starts in the central region, then passes through some rabbit region and continues outward, crossing many components.



Conjecture. For any fixed q, as we follow any vein, the period q tessellation, "restricted" to each corresponding zero-kneading region, remains "isomorphic" except when we cross into a component of period q. Then it becomes "more complicated".

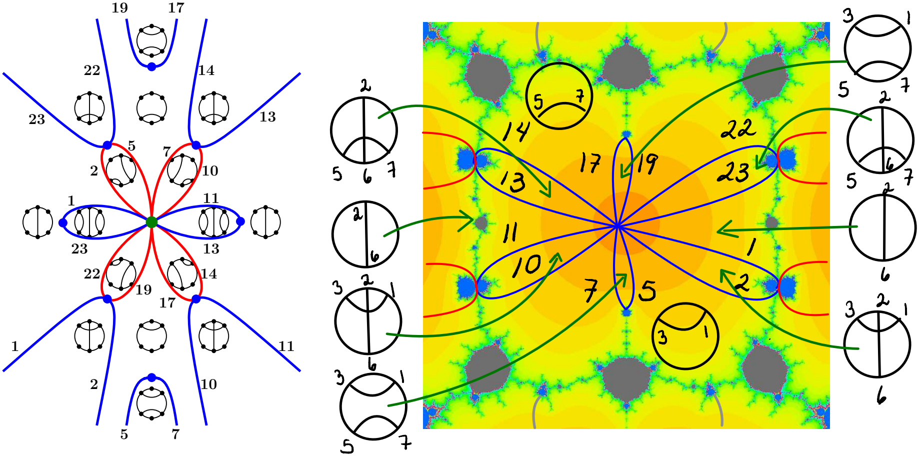
The orbit portraits associated with a tessellation will be considered as an essential part of the tessellation.

Let $\mathcal{E} \subset \mathcal{S}_p$ and $\mathcal{E}' \subset \mathcal{S}_{p'}$ be two escape regions.

Definition. $\mathrm{Tes}_q(\mathcal{E})$ is **isomorphic** to $\mathrm{Tes}_q(\mathcal{E}')$ if:

There is a one-to-one correspondence between the faces of $\mathrm{Tes}_q(\mathcal{S}_p)$ intersecting $\mathcal{E}$ and the faces of the $\mathrm{Tes}_q(\mathcal{S}_{p'})$ intersecting $\mathcal{E}'$, preserving orbit portraits, and preserving the angles of the parameter rays within $\mathcal{E}$ or $\mathcal{E}'$ which lie on the boundary of each such face.

Example: Tes_2 for Basilica and Airplane



The outer part of the left hand figure represents the basilica region of $\mathcal{S}_2$. $\mathrm{Tes}_2(\text{basilica})$ is **isomorphic** to $\mathrm{Tes}_2(\text{airplane})$.

(Denominators: 8 for dynamic angles, 24 for parameter angles.)

The Vein Conjecture: "More Complicated".

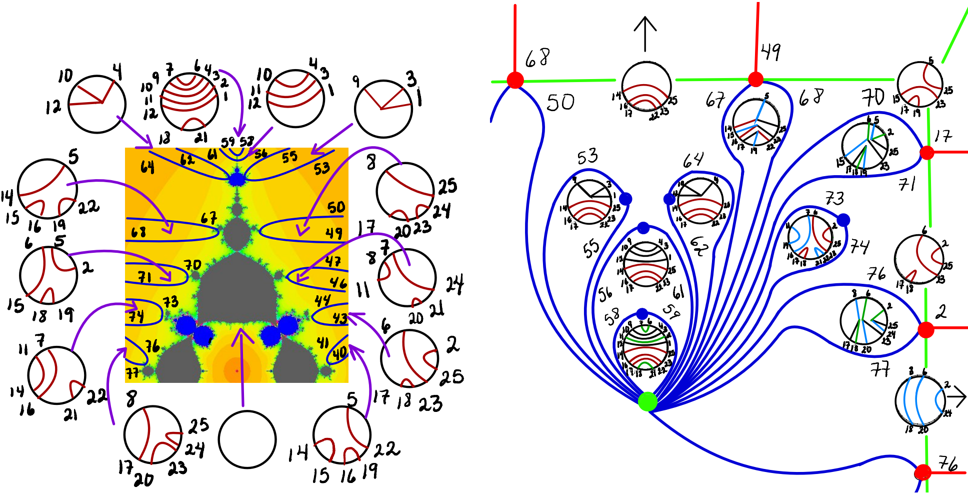
Again consider two escape regions $\mathcal{E} \subset \mathcal{S}_p$ and $\mathcal{E}' \subset \mathcal{S}_{p'}$.

Definition. $\mathrm{Tes}_q(\mathcal{E}) << \mathrm{Tes}_q(\mathcal{E}')$ if:

*$\mathrm{Tes}_q(\mathcal{E}')$ has **more** faces than $\mathrm{Tes}_q(\mathcal{E})$, and each face of $\mathrm{Tes}_q(\mathcal{E}')$ is a subset of some face of $\mathrm{Tes}_q(\mathcal{E})$.*

*Furthermore the orbit portrait for each face of $\mathrm{Tes}_q(\mathcal{E}')$ is **bigger** than the orbit portrait for the corresponding face of $\mathrm{Tes}_q(\mathcal{E})$.*

Example: Tes$_3$ for basilica and airplane.



(Denominators 26, 78.) The unique shared face on the left has trivial orbit portrait. The twelve on the right are all non-trivial. Between rays 67 and 68 on the left, the orbit portrait has three simple arcs. On the right it has three tripods.

The Similarity Conjecture

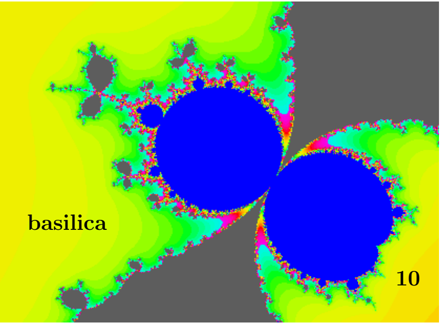
Imitating Douady and Hubbard, a map in $\mathcal{S}_p$ will be called a **Misiurewicz map** if the free critical point $-a$ is eventually periodic repelling.

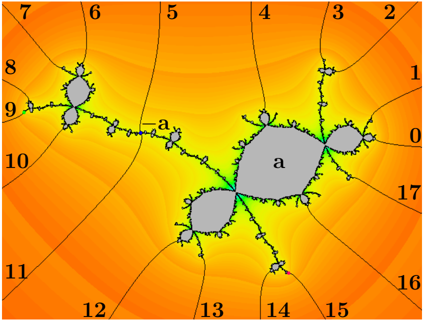
Tan Lei proved the following:

> *If $f(z) = z^2 + c$ is a quadratic Misiurewicz map, then under iterated magnification, the parameter plane near f looks more and more like the dynamic plane near c (up to a fixed scale change).*

Conjecture. For a Misiurewicz map $F \in \mathcal{S}_p$, under iterated magnification, the parameter space near F looks more and more like the dynamic plane near $2a_F$ (up to a fixed scale change and rotation).

Similarity Example (A Chebyshev map in $\mathcal{S}_2$).





On the left: a copy of $\mathbb{M}$ in $\mathcal{S}_2$. The Chebyshev point at the left tip of this copy, is the landing point of the $17/18$ parameter ray.

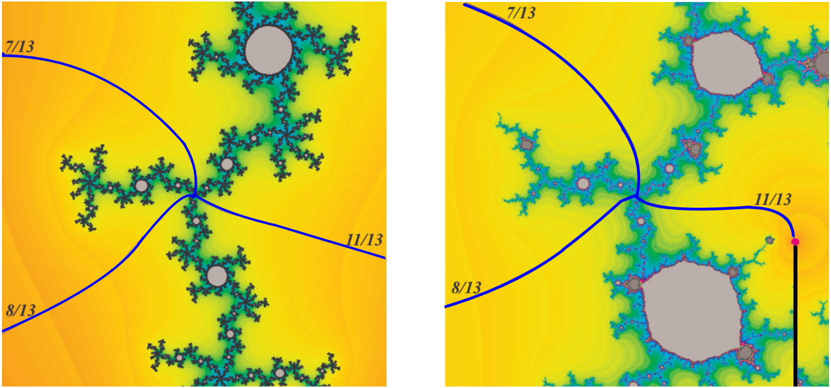
On the right: Julia set for this Chebyshev point.

Note that $\{5, 11, 17\} \mapsto 15 \mapsto 9 \pmod{18}$.

Here $2a$ is at the landing point of the $17/18$ ray.

The $9/18 = 1/2$ ray is fixed under tripling.

Similarity Example (between Kokopelli and 0010).



On the left: Julia set picture centered at $2a$ for a Misiurewicz map $F_0 \in \mathcal{S}_4$. In this example, $2a$ is a fixed point of rotation number $1/3$. On the right: Corresponding parameter space picture, centered at F_0 and suitably rotated and magnified, with the Kokopelli region to the left and a 0010 region to the right.

Canonical Coordinates.

Let $\mathcal{S} \subset \mathbb{C}^2$ be an arbitrary smooth affine curve, defined by a polynomial equation $\Phi(z, w) = 0$.

Then there is a canonical closed 1-form on $\mathcal{S}$,

$$\Phi_z dw + \Phi_w dz \ .$$

Near any point of $\mathcal{S}$ we can integrate this 1-form to obtain a **canonical coordinate** g,

well defined up to an additive constant,

which maps a neighborhood biholomorphically into $\mathbb{C}$.

But in general g cannot be extended to a global coordinate.

Zero-Kneading Case:

$\mathcal{E}$ corresponds to a neighborhood of infinity.

Non-Zero Kneading:

The puncture point maps to the finite plane, and $\mathcal{E}$ is locally a branched covering of the canonical plane.

THE END!

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